Interacting Hopf Algebras

the theory of linear systems

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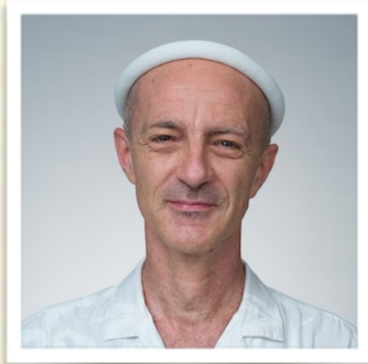

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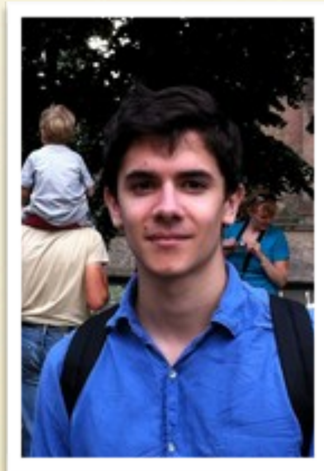

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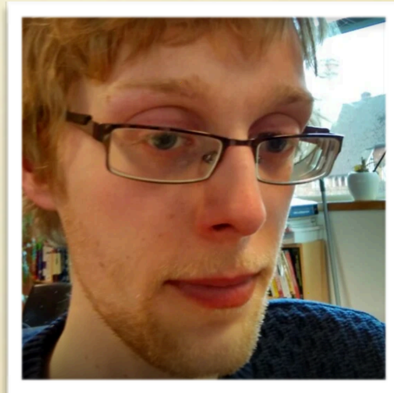

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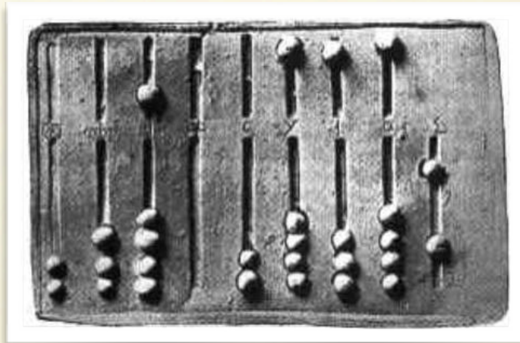
I. Interacting Hopf Algebras

II. Controversial Intermezzo

III. Ramifications

Numeral Systems

Roman numbers are quite inconvenient to perform the elementary arithmetic operations



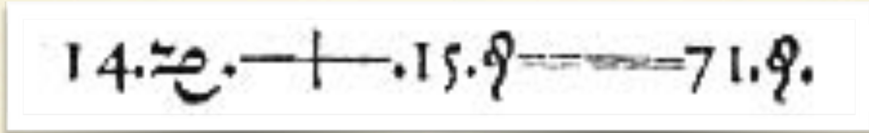
In *Liber Abbaci* (1201), Fibonacci introduce in Europe the Arab numeral system together with the basic algorithm

In1280, the city of Florence forbade the use of Arab *ciphers*

Nowadays, the introduction of Arab numbers is considered a fundamental moment in the History of Science

Equations

In *The Whetstone of Witte* (1557), Robert Recorde introduced the symbol =



$$14x + 15 = 71$$

Sign	Wording	Meaning
φ	dragma, numerus	the number, the absolute term
ꝛe	radix	the root, the unknown, x
ʒ	zensus (census)	the square, x^2
ce	cubus	the cube, x^3
ʒʒ	zensdezens	the fourth power, x^4
β	sursolidum	the fifth power, x^5
ʒce	zensicubus	the sixth power, x^6
bβ	bissursolidum	the seventh power, x^7
ʒʒʒ	zenszensdezens	the eighth power, x^8
cce	cubus de cubo	the ninth power, x^9

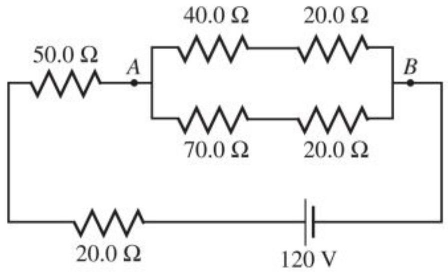
Polynomials

Our notation for polynomials was introduced by Descartes in *La Géometrie* (1637)

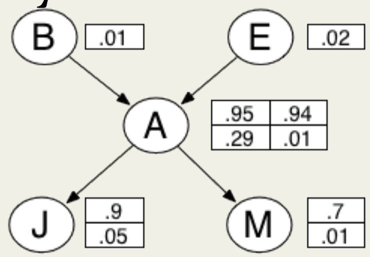
After we started writing x^2, x^3,... x^n, we could think about x^{-2} or $x^{0.5}$

Network diagrams

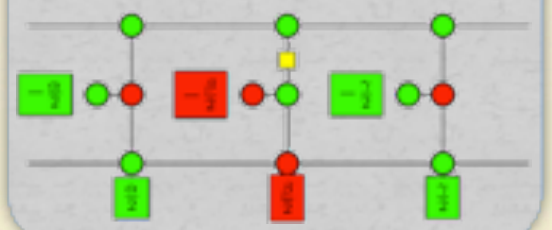
Electrical Circuits



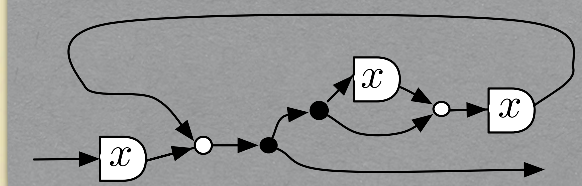
Bayesian Networks



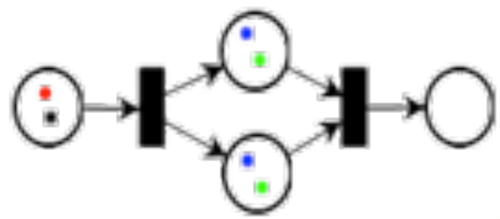
Quantum Processes



Signal Flow Graphs

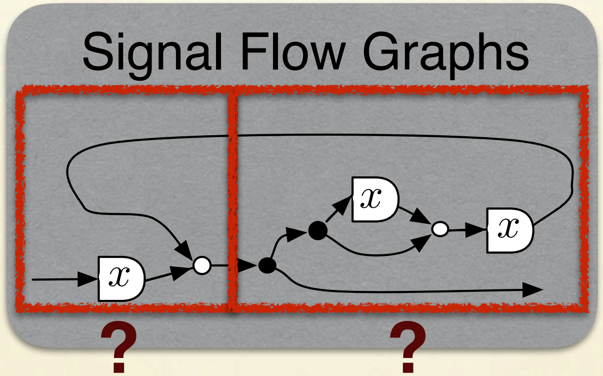


Petri Nets



Compositional Semantics

Diagrammatic languages are not really made of syntax.



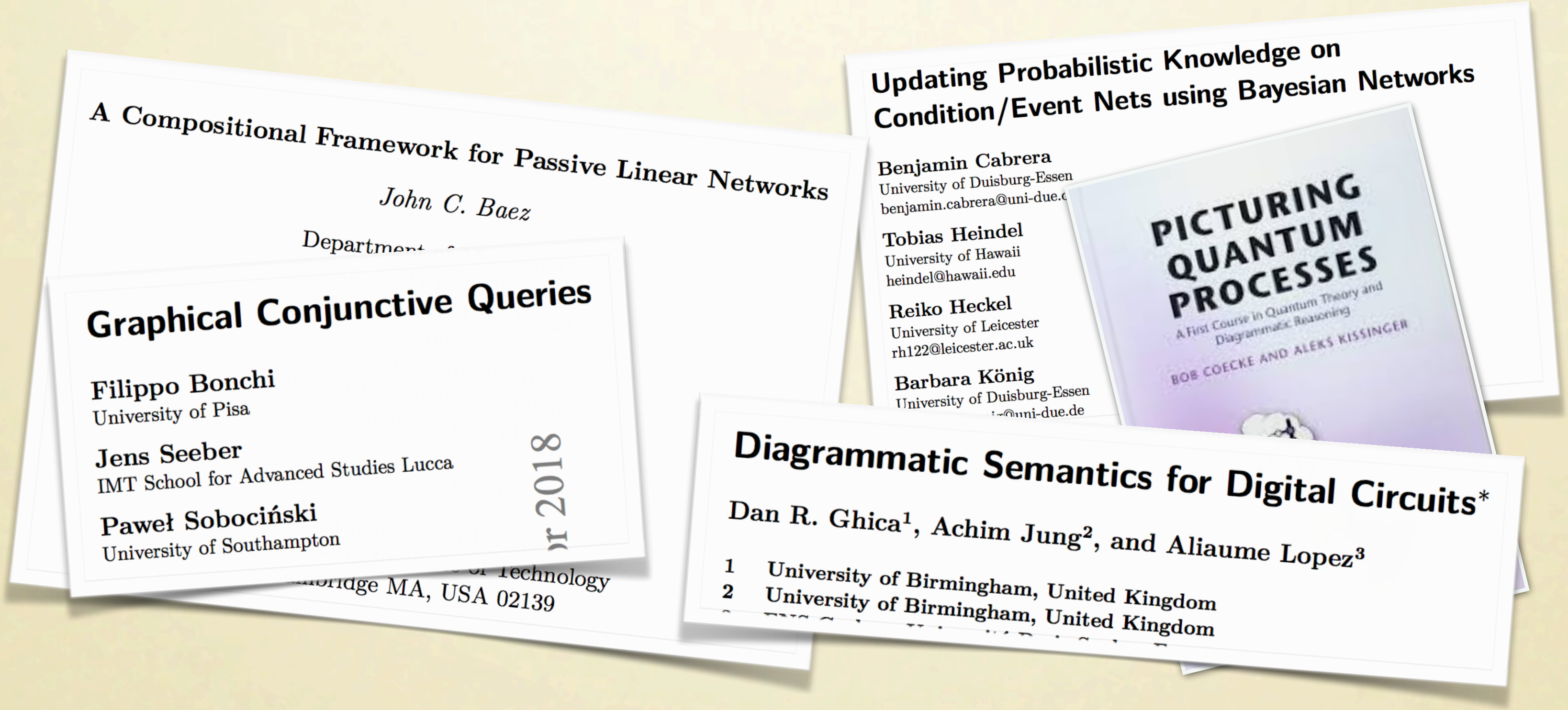
We are able to describe the behaviour of the whole systems

But not the behaviour of the single components

The behaviour of the whole system should be "reducible" to the behaviour of its components

Compositional Modelling

There is an emerging, multi-disciplinary field aiming at studying different sorts of networks **compositionally**, inspired by the **algebraic methods** of programming language semantics.



Diagrams are first-class citizens of the theory. The appropriate algebraic setting is **monoidal** (and not **cartesian**) categories.

More and more influential

The Future Will Be Formulated Using Category Theory

Jayshree Pandya Contributor
COGNITIVE WORLD Contributor Group ⓘ
AI & Big Data
Jayshree Pandya is Founder of Risk Group & Host of Risk Roundup.

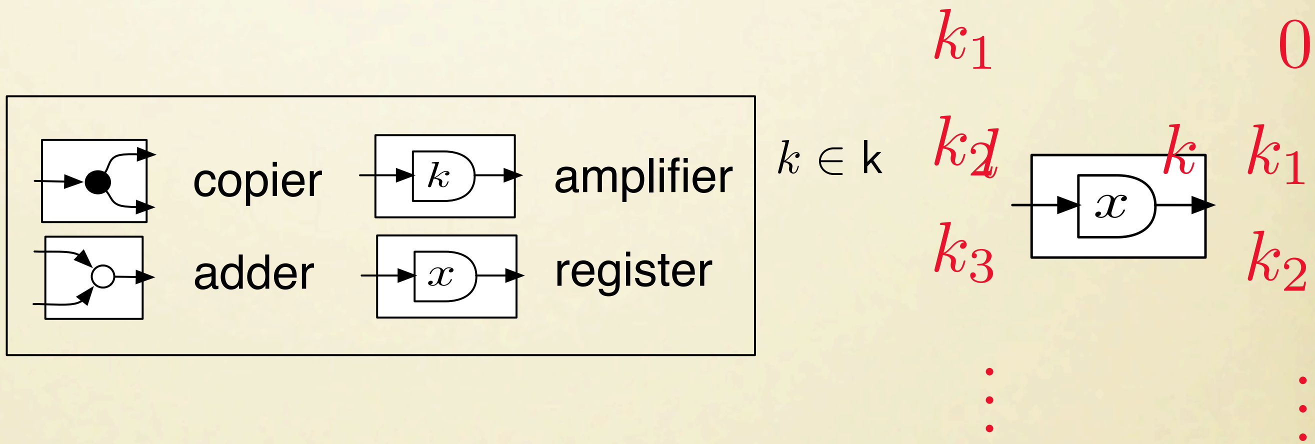
A new approach to defining and designing systems is coming.

https://www.forbes.com/sites/cognitiveworld/2019/07/29/the-future-will-be-formulated-using-category-theory/#71a09469625e

I. Interacting Hopf Algebras

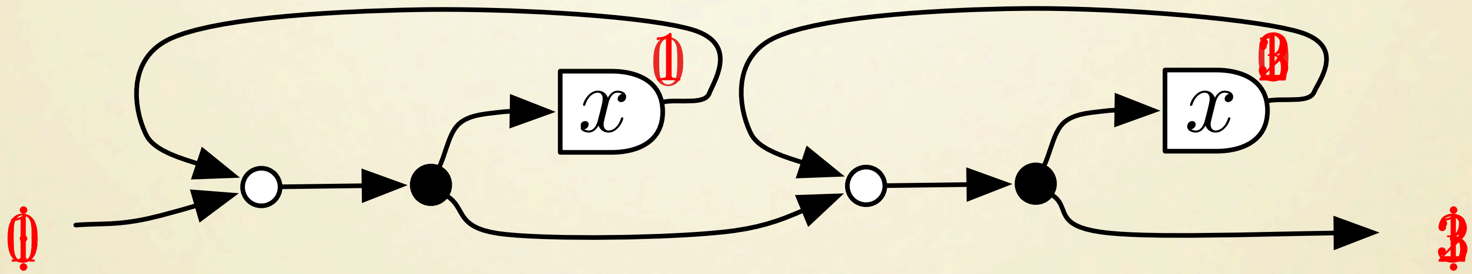
Signal Flow Graphs

Signal Flow Graphs are **stream** processing circuits widely adopted in Control Theory and Signal Processing



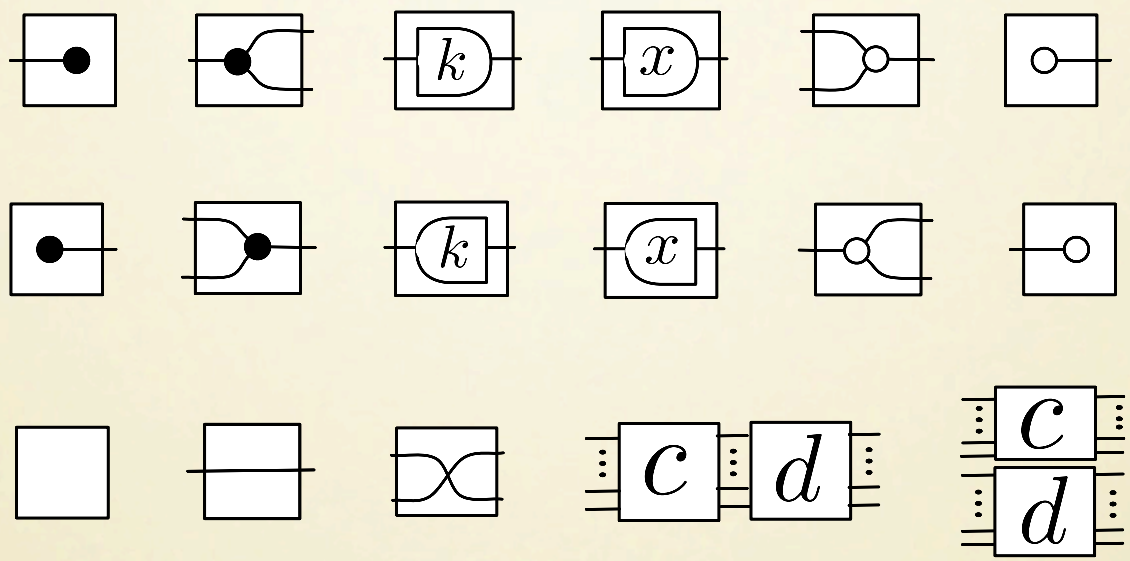
Claude Shannon. *The theory and design of linear differential equation machines* (1942).

Signal Flow Graphs

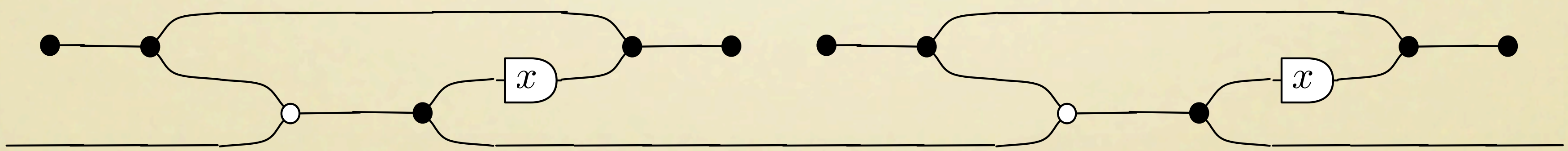


String Diagrammatic Syntax

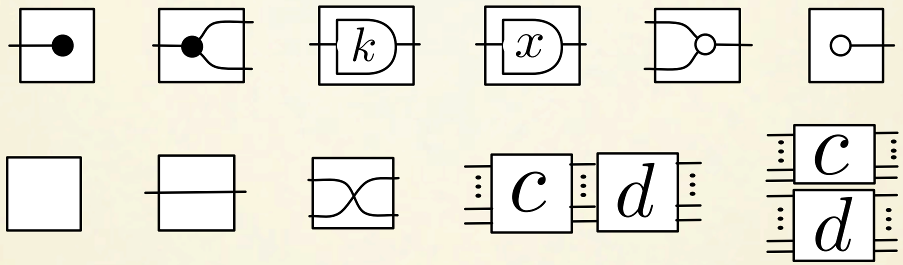
$c, d ::=$



Subject to the laws of PROPs



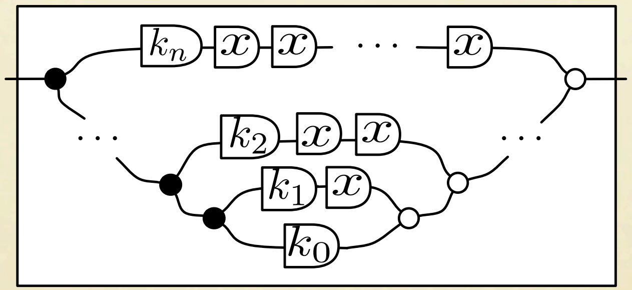
Functional Circuits

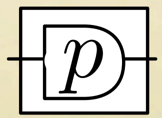
$c, d ::=$ 

We can represent polynomial

$$p = k_0 + k_1 x + \cdots + k_n x^n$$

as



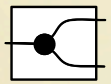
Hereafter denoted by 

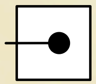
Polynomial Matrices

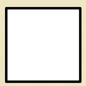
The denotational semantics of functional circuits is given by the PROP morphism

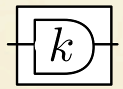
$$\overrightarrow{[\![\cdot]\!]} : \overrightarrow{\mathsf{Circ}} \to \mathsf{Mat}_{k[x]}$$

defined inductively as follows

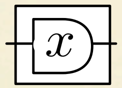
 $\longmapsto \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

 $\longmapsto \; !$

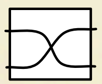
 $\longmapsto id_0$

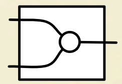


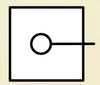
$\longmapsto \begin{pmatrix} k \end{pmatrix}$

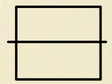


$\longmapsto \begin{pmatrix} x \end{pmatrix}$

 $\longmapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

 $\longmapsto \begin{pmatrix} 1 & 1 \end{pmatrix}$

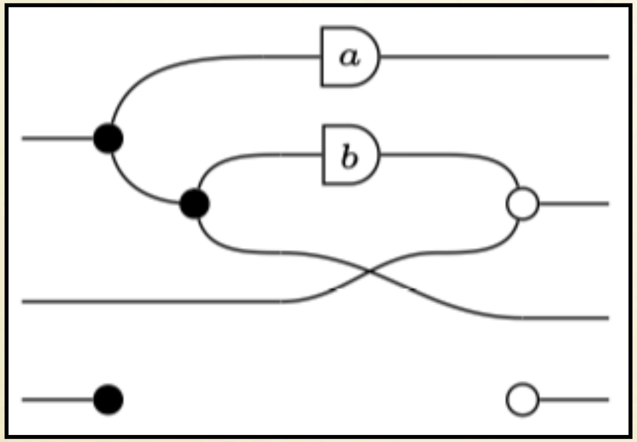
 $\longmapsto \; ?$

 $\longmapsto id_1$

$$c_1 \oplus c_2 \longmapsto \overrightarrow{[\![c_1]\!]} \oplus \overrightarrow{[\![c_2]\!]} \qquad c_1 \,;\, c_2 \longmapsto \overrightarrow{[\![c_1]\!]} \,;\, \overrightarrow{[\![c_2]\!]}$$

where $!$ and $?$ are the unique morphism given by initiality and finality of 0

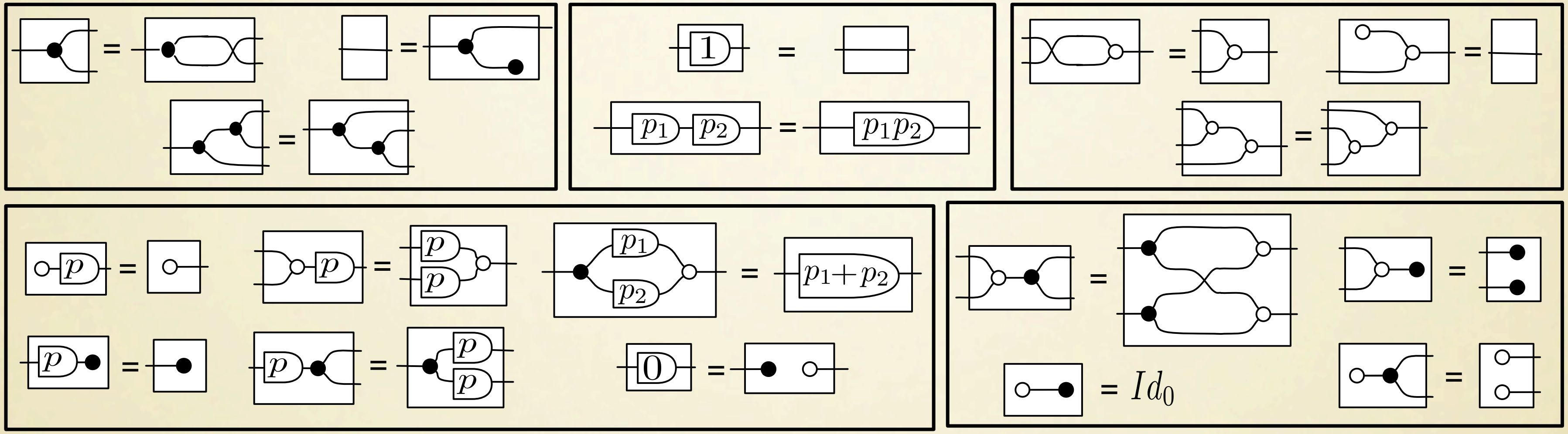
Example



$$\longmapsto \begin{pmatrix} a & 0 & 0 \\ b & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Hopf Algebras

The theory $\mathbb{HA}$ is $\overrightarrow{\mathsf{Circ}}$ quotiented by the following axioms



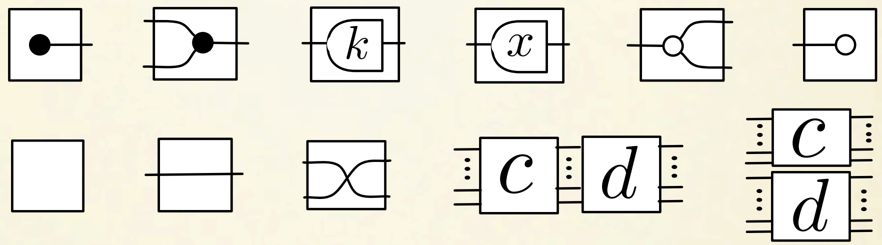
Soundness and Completeness

$\mathbb{HA}$ is isomorphic to $\mathsf{Mat}_{\mathsf{k}[x]}$

$$\overrightarrow{[\![c]\!]} = \overrightarrow{[\![d]\!]} \iff c \stackrel{\mathbb{HA}}{=} d$$

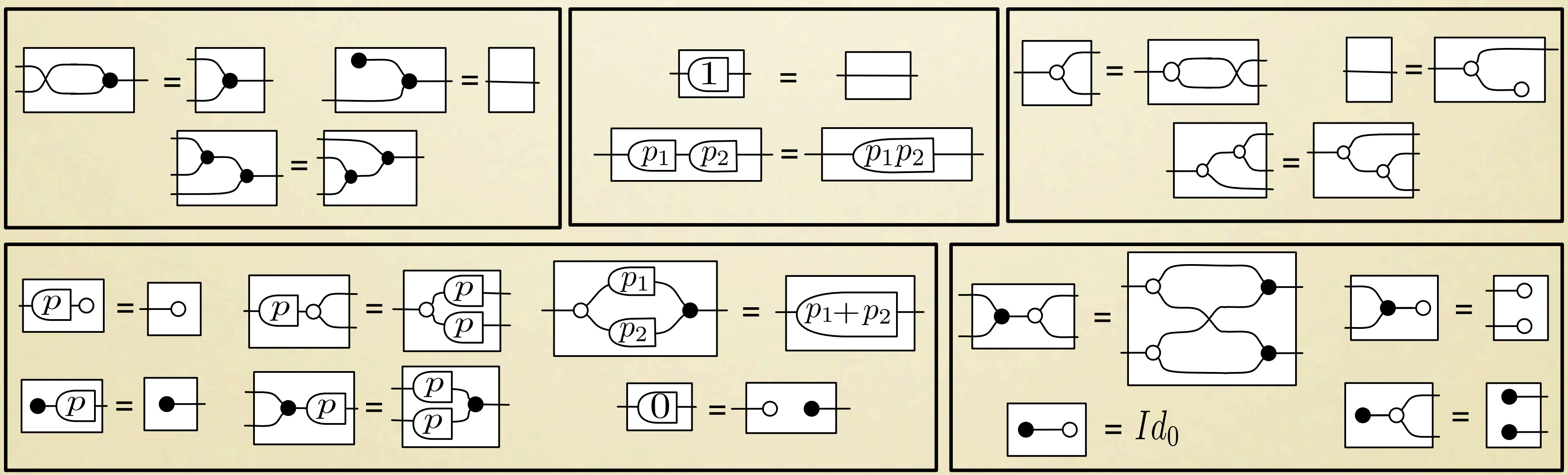
Cofunctional Circuits

$c, d ::=$



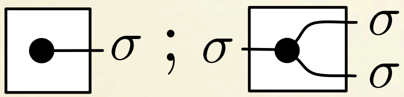
The denotational semantics $\overleftarrow{[\![\cdot]\!]} : \overleftarrow{\mathsf{Circ}} \to \mathsf{Mat}^{op}_{k[x]}$ is given by duality

The theory $\mathbb{HA}^{op}$ is $\overleftarrow{\mathsf{Circ}}$ quotiented by the following axioms



Semantics of Generalised Circuits

When we allow combinations of functional and co-functional circuits (like in feedbacks) we may get *relational* behaviours:

For instance,  expresses the diagonal relation

Moreover, polynomials are not enough: we need *fractions* of polynomials.

$$\mathsf{k}[x] \mapsto \mathsf{k}(x) \mapsto \mathsf{k}((x))$$

polynomials

fractions of polynomials

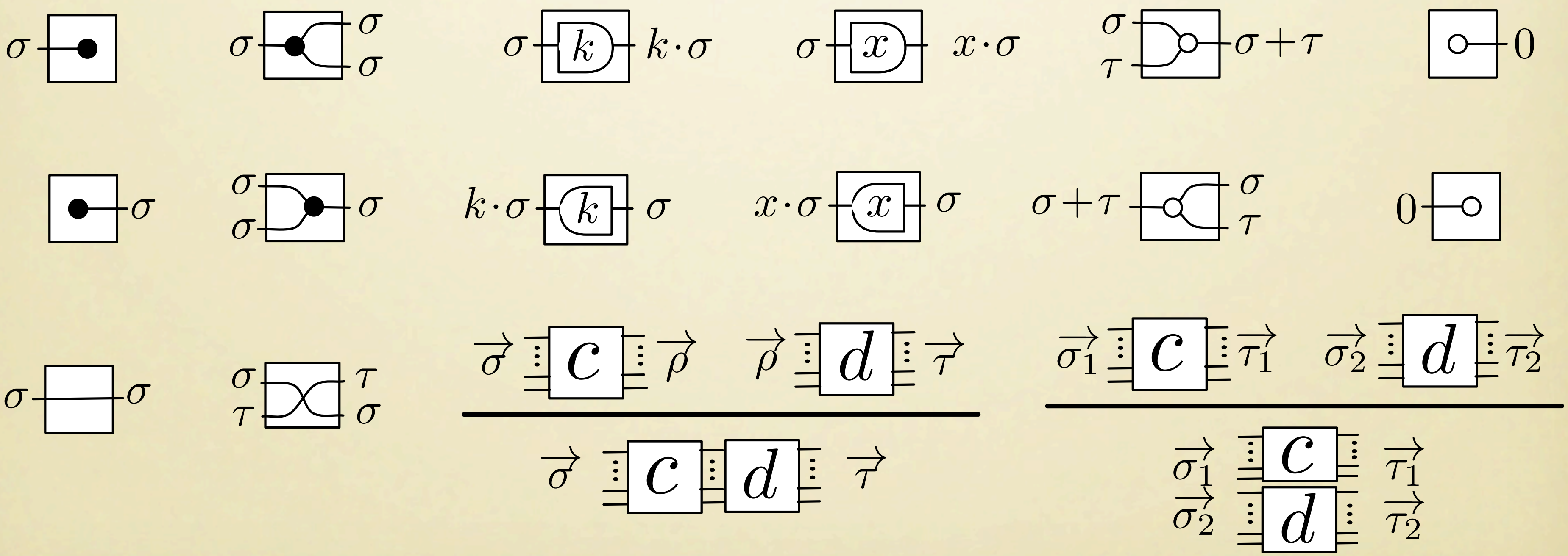
Laurent series

Semantics of Generalised Circuits

The denotational semantics of circuits is given by the PROP morphism

$$[\![\cdot]\!] : \mathsf{Circ} \to \mathsf{LinRel}_{\mathsf{k}((x))}$$

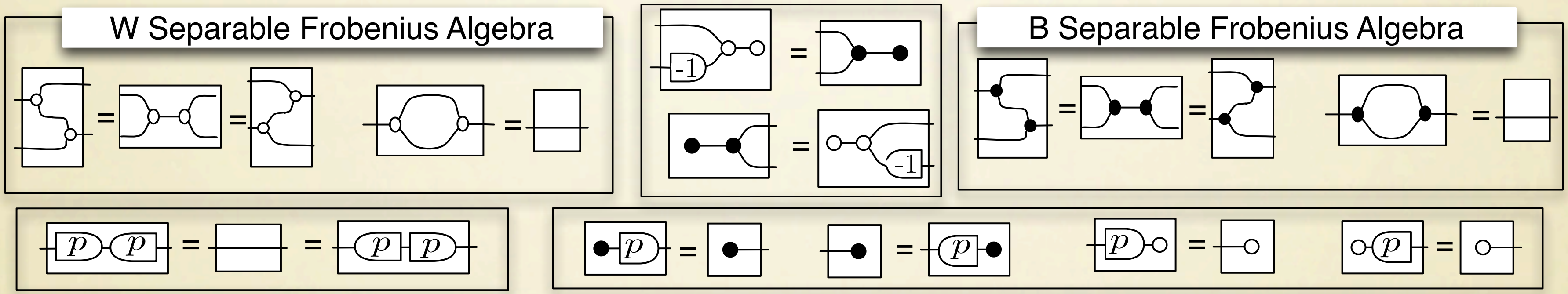
defined as follows



Interacting Hopf Algebras

The theory $\mathbb{IH}$ is Circ quotiented by the axioms of $\mathbb{HA}$, $\mathbb{HA}^{op}$

and the following



Soundness and Completeness

The proof exploits a technique introduced by Steve Lack in *Composing PROPs* (2004)

$$[\![c]\!] = [\![d]\!] \Longleftrightarrow c \stackrel{\mathbb{IH}}{=} d$$

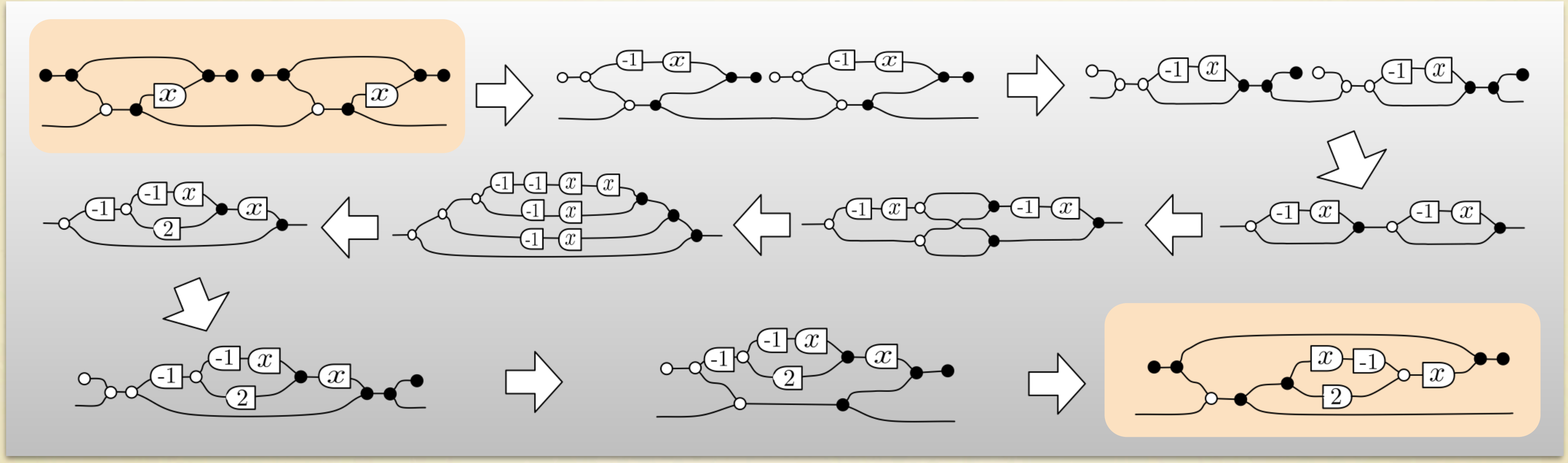
Bonchi, Sobocinski, Zanasi
Interacting Hopf Algebras
Journal of Pure and Applied Algebra (2017)

Kleene's Theorem

$\mathbb{IH}$ is isomorphic to $\mathsf{LinRel}_{\mathsf{k}(x)}$

Equational Reasoning

Proof that two diagrams represent the same dynamical system:



Actually, this holds in general:

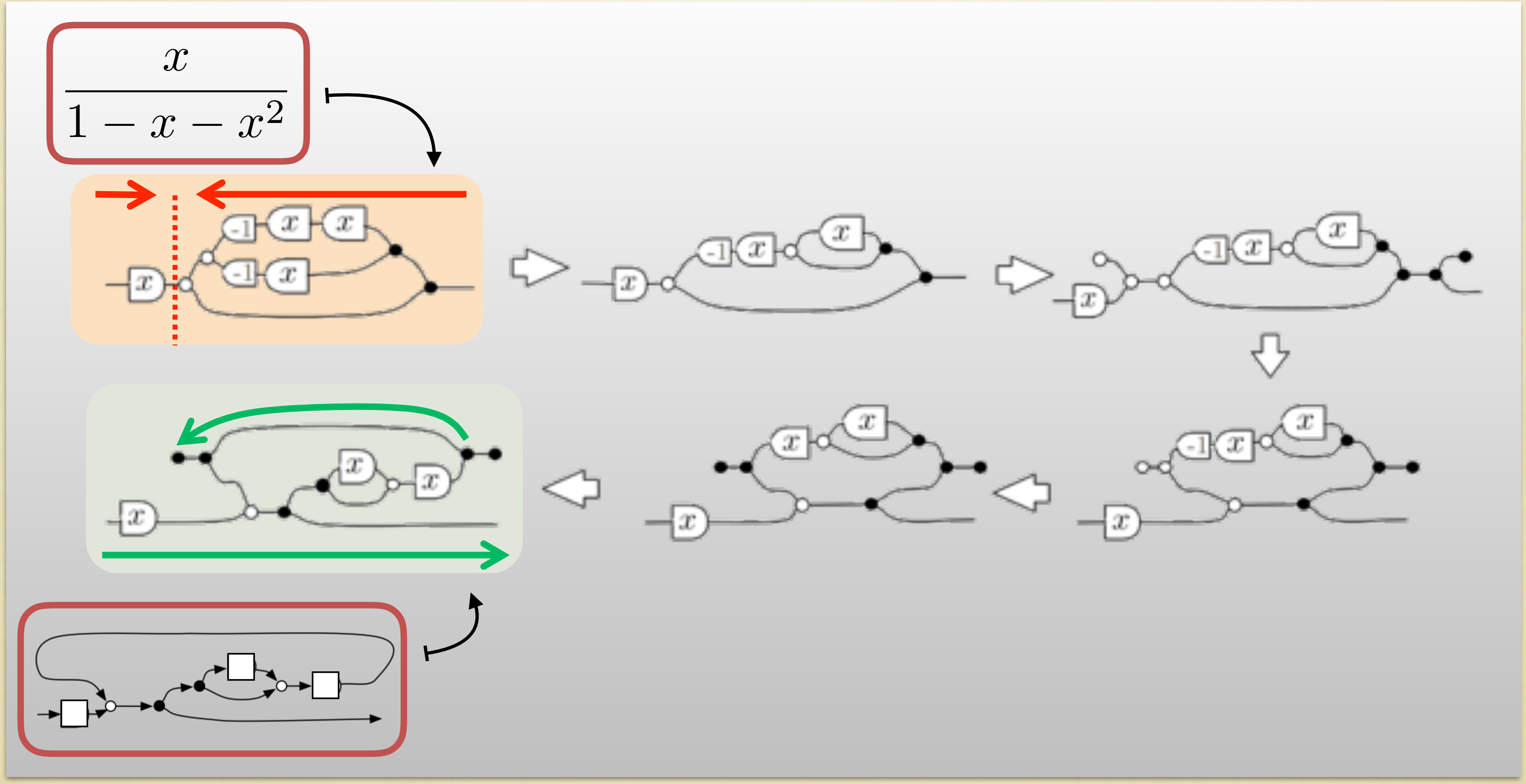
Normal Form

any circuits is equivalent to one in $\overrightarrow{\mathsf{Circ}}$ with a feedback loop

The analogue of Bohm-Jacopini's Theorem

Equational Reasoning

From a specification (a rational fraction) to its implementation as a linear dynamical system.



II. Controversial Intermezzo

Flow direction is FUNDAMENTAL

“flow graphs differ from electrical network graphs in that their branches are directed. In accounting for branch directions it is necessary to take an entirely different line of approach from that adopted in electrical network topology.”

S.J. Mason. *Feedback Theory: I. Some Properties of Signal Flow Graphs.* 1953

Flow direction is EVIL

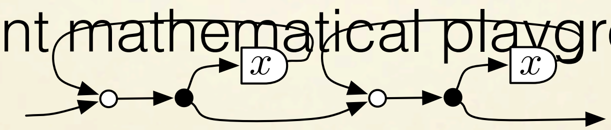
"Adding a signal flow direction is often a figment of one's imagination, and when something is not real, it will turn out to be cumbersome sooner or later. [...] The input/output framework is totally inappropriate for dealing with all but the most special system interconnections. [The input/output representation] often needlessly complicates matters, mathematically and conceptually. A good theory of systems takes the behavior as the basic notion."

J. Willems. *Linear systems in discrete time.* 2009

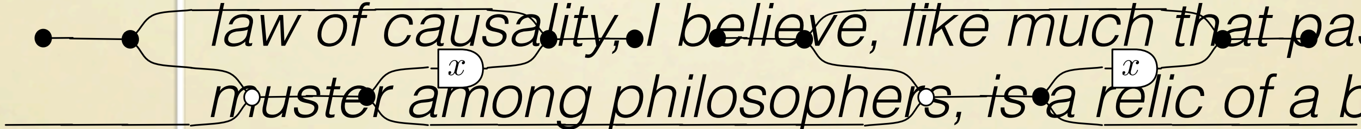
Flow directionality

Circ does not rely on flow directionality as a primitive

This allows for a more flexible syntax, disclosing a rich and elegant mathematical playground: $\mathbb{IH}$

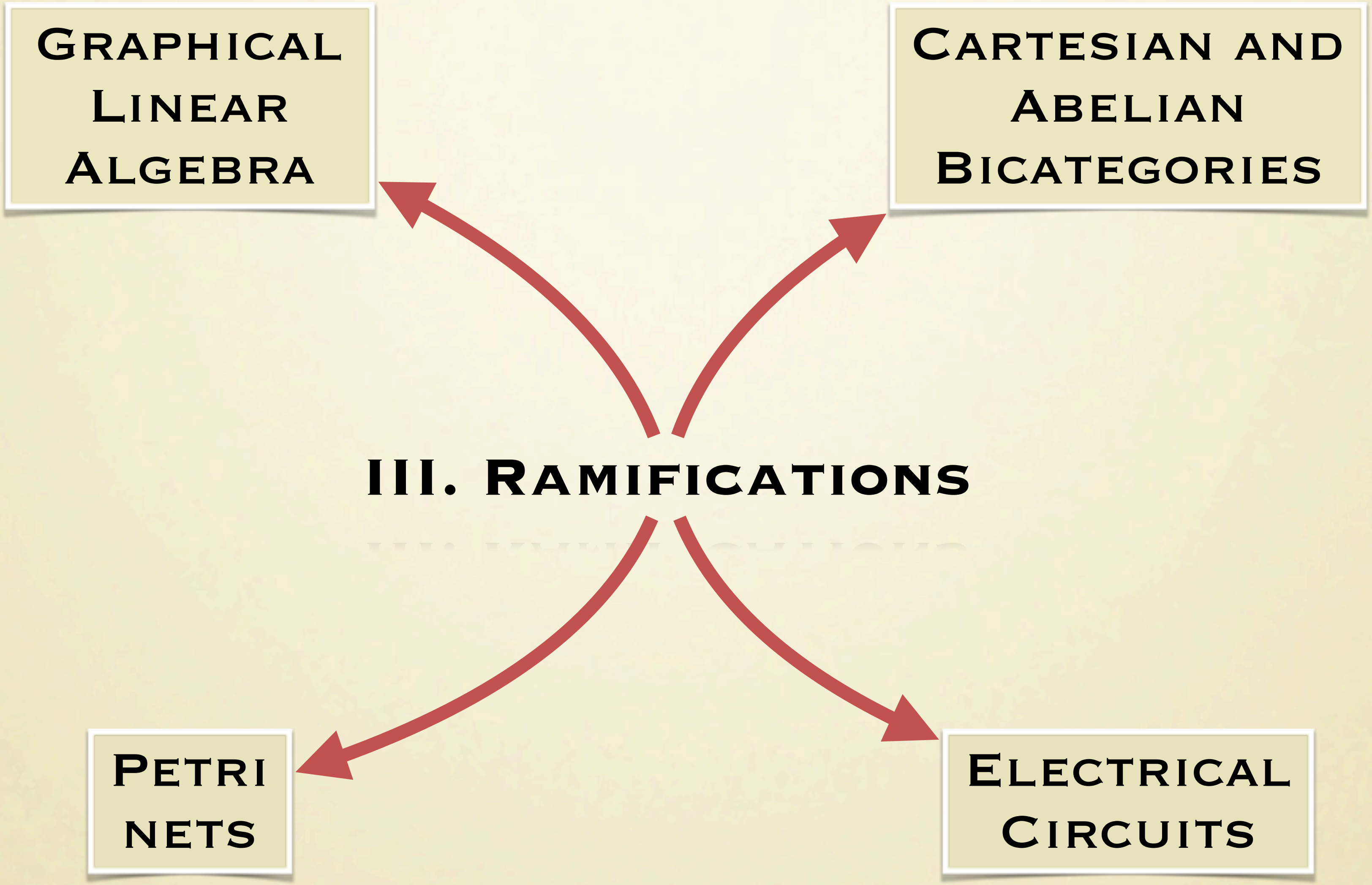


Whenever flow directionality matters, we can always rewrite any circuit in its normal form



"The reason why physics has ceased to look for causes is that in fact there are no such things. The law of causality, I believe, like much that passes muster among philosophers, is a relic of a bygone age, surviving, like the monarchy, only because it is erroneously supposed to do no harm."

(Bertrand Russel - 1913)


Graphical Linear Algebra
Cartesian and Abelian Bicategories
III. Ramifications
Petri nets
Electrical Circuits

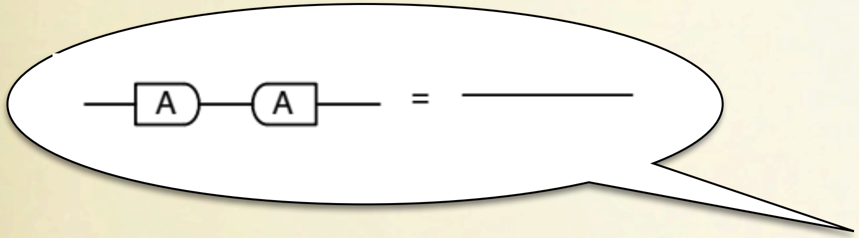
Graphical Linear Algebra

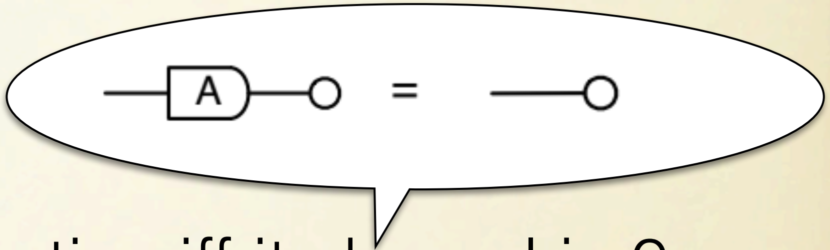
For dynamical systems, we need the field of fractions of polynomials, but actually the theorem holds for any field

This allows to *re*prove
well-known theorems of linear algebra
by means of our axioms

You can find many examples in the blog:
https://graphicallinearalgebra.net

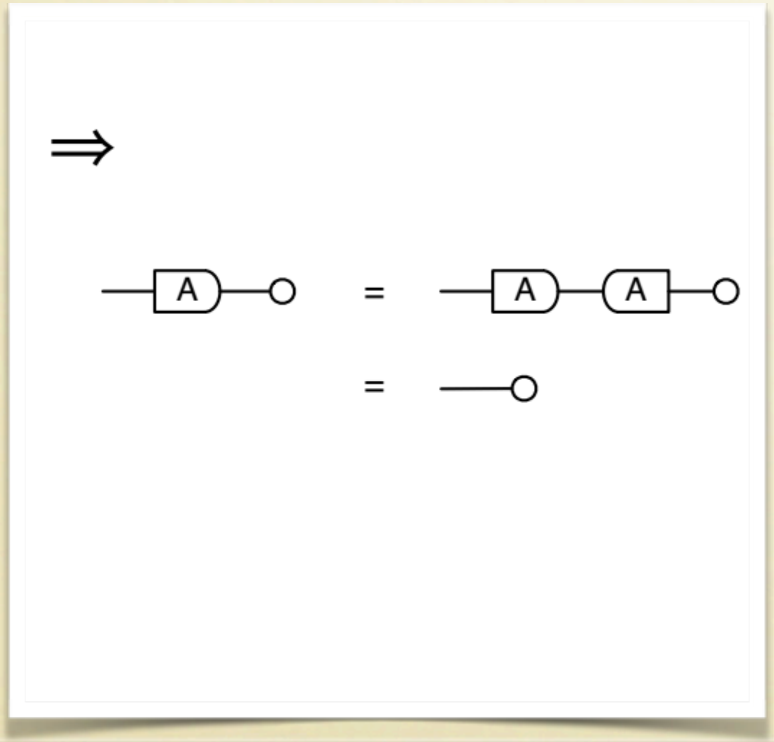
Equational Proof

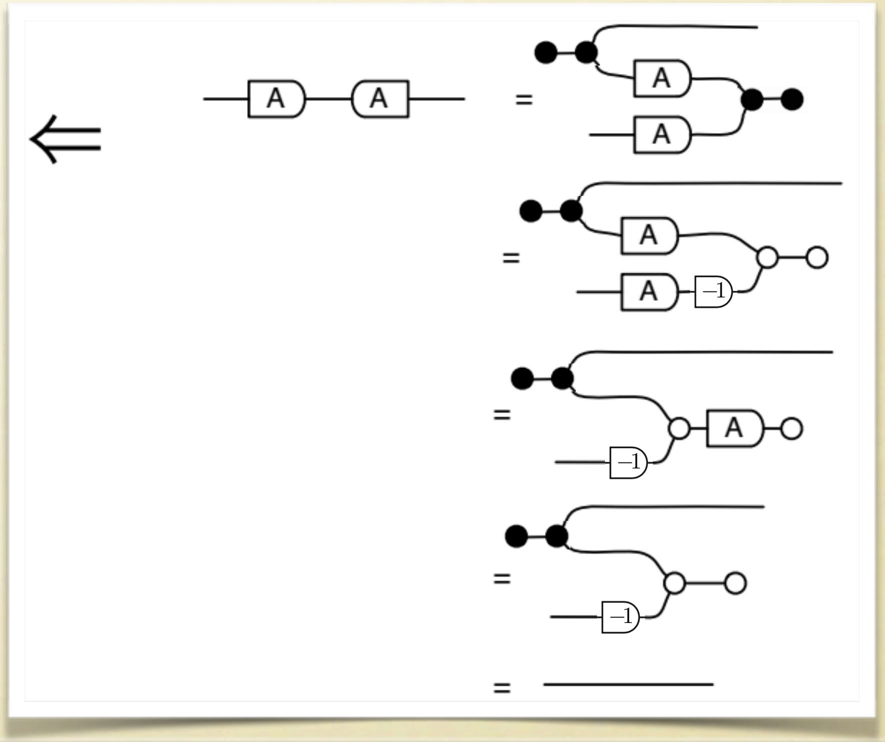


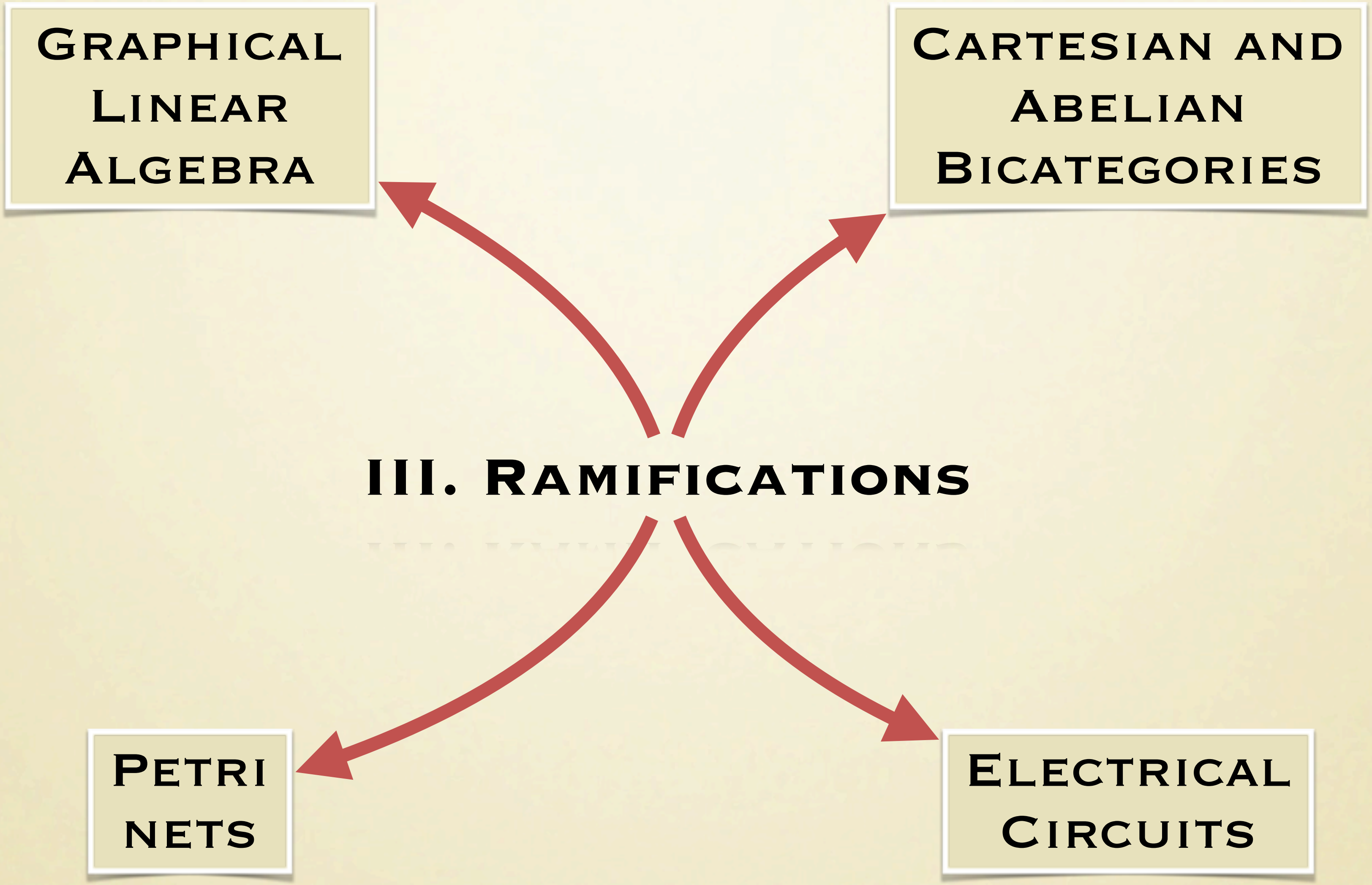


Proposition: a matrix A is injective iff its kernel is 0.

Proof



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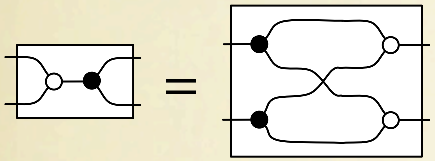
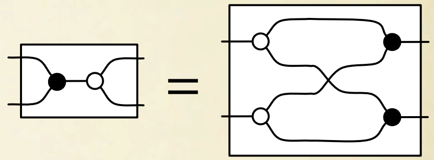
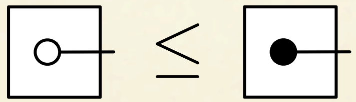
Cartesian and Abelian Bicategories



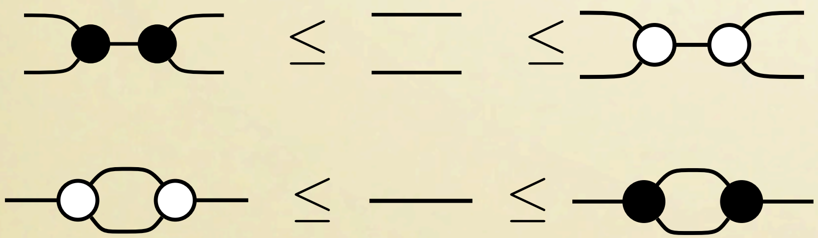


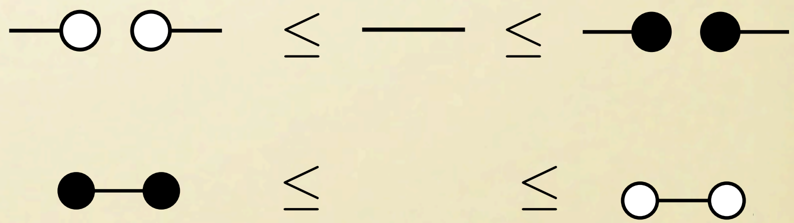
Switching black and white colouring does not change the axiomatisation

To axiomatise the inclusion relation is enough to add

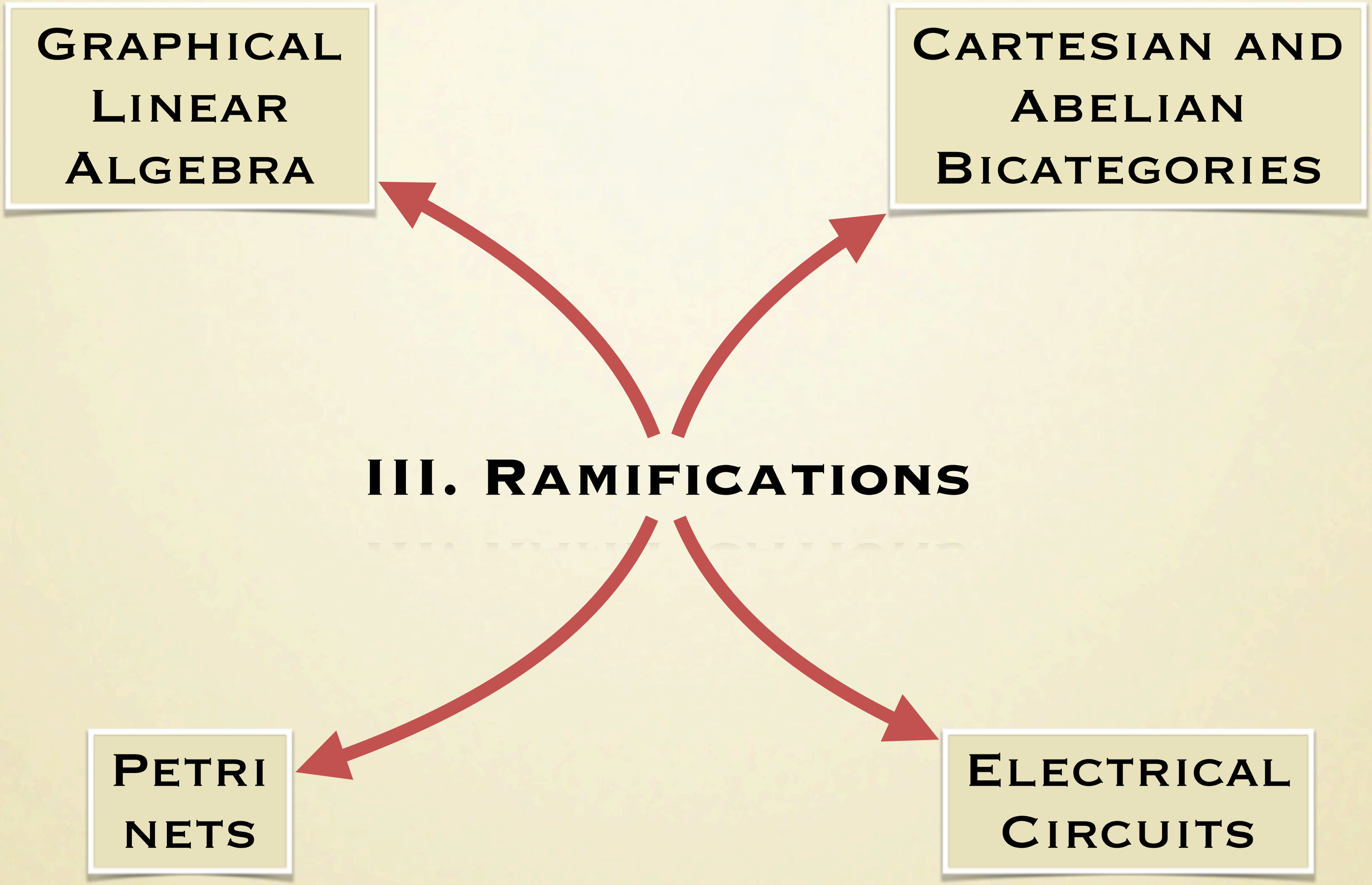


It turns out that $\mathbb{IH}$ is an Abelian bicategory (Carboni-Walters)





Bonchi, Holland, Pavlovic, Sobocinski. *Refinement for Signal Flow Graphs*. CONCUR 2017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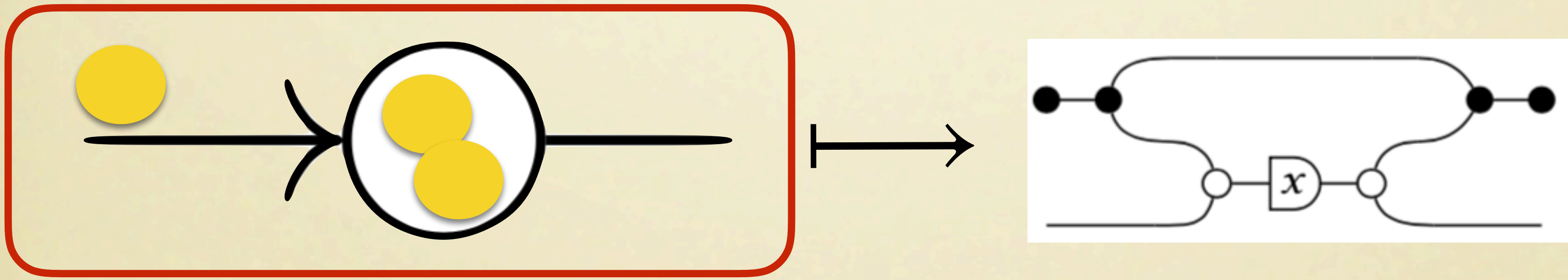
Petri Nets

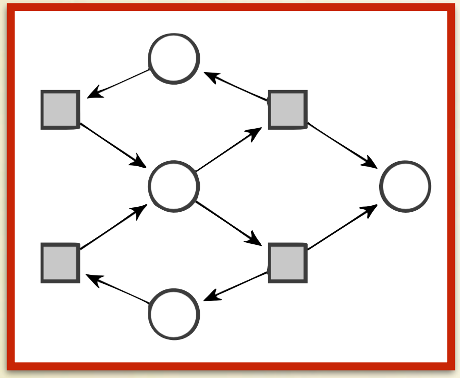
Instead of a field, pick a semiring without additive inverses: the set $\mathbb{N}$ of natural numbers.

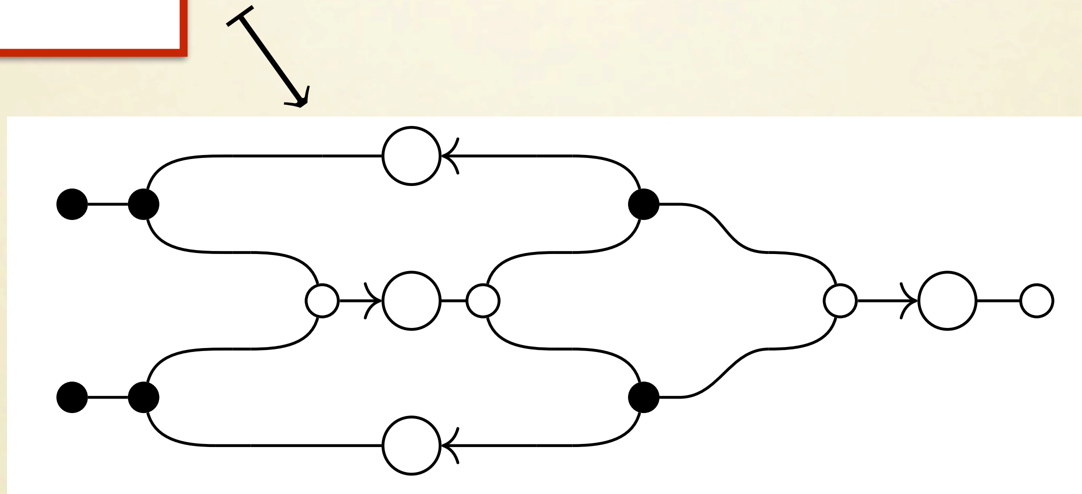
Because $\mathbb{N}$ does not have additive inverses, it is suitable to model **resources** in a distributed system.

We interpret the **place of a Petri net** as a $\mathbb{N}$-circuit diagram

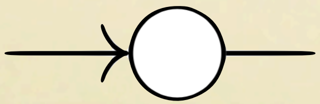
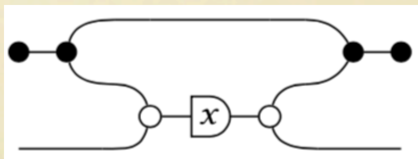


Petri Nets



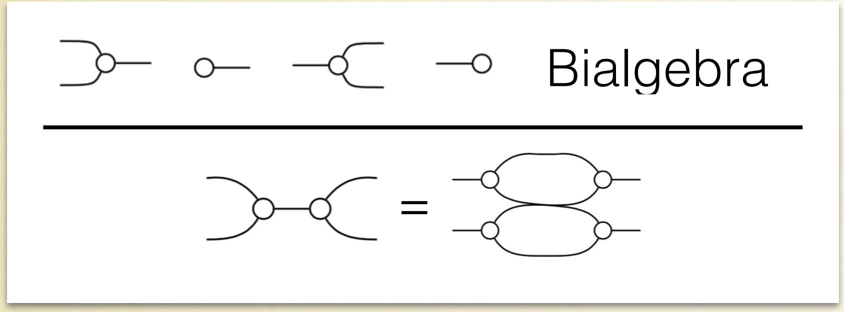


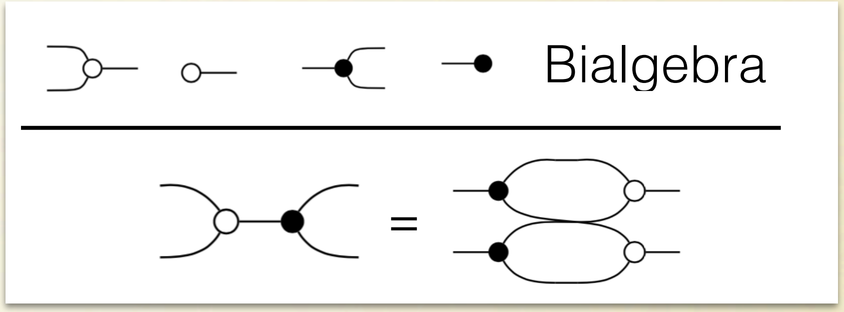
dove

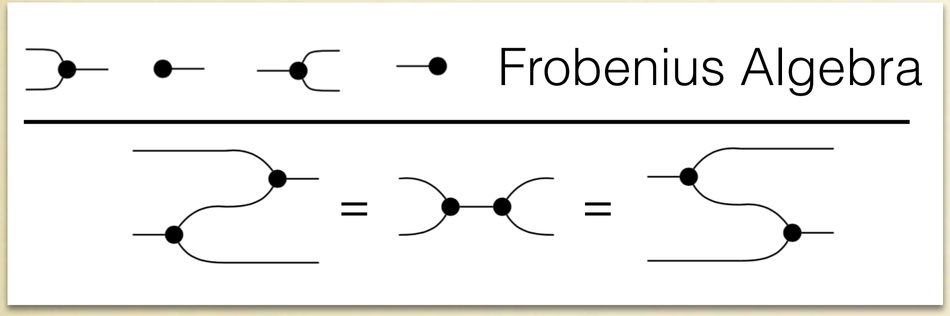
 ::= 

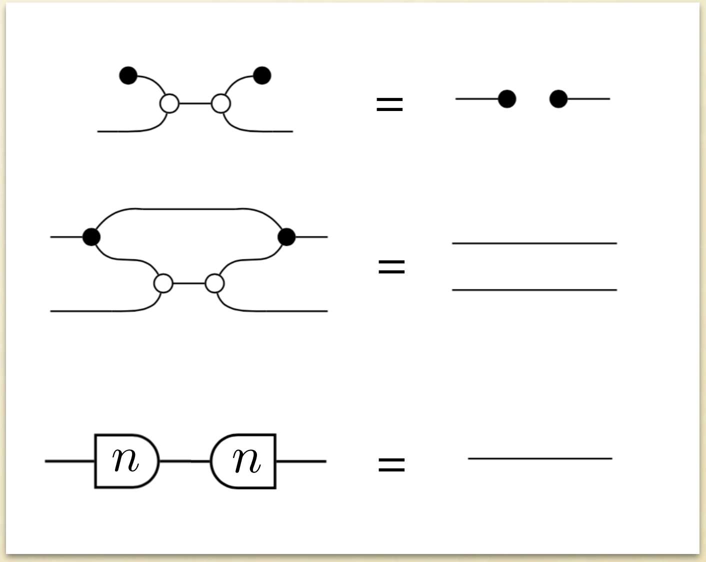
Equational Theory

$\mathbb{A}\mathbb{R}$: Algebra of Resources









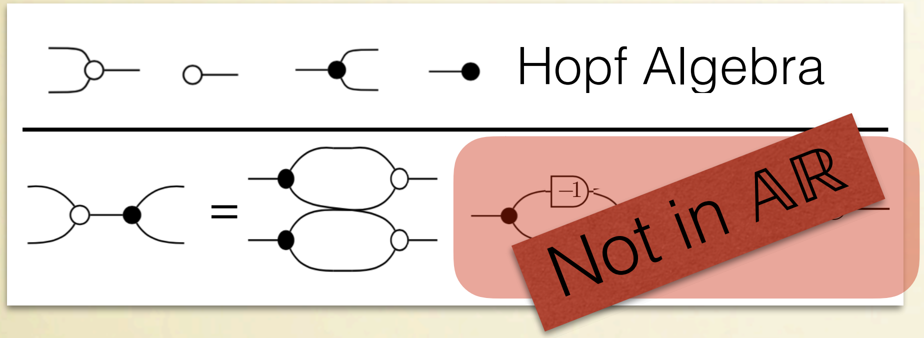
$n \neq 0$

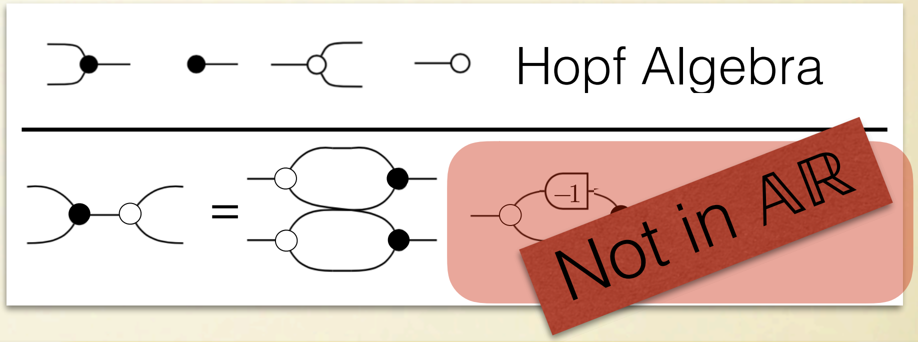
Bonchi, Holland, Piedeleu, Sobocinski, Zanasi - *Diagrammatic Algebra: From Linear to Concurrent Systems*, POPL, 2019.

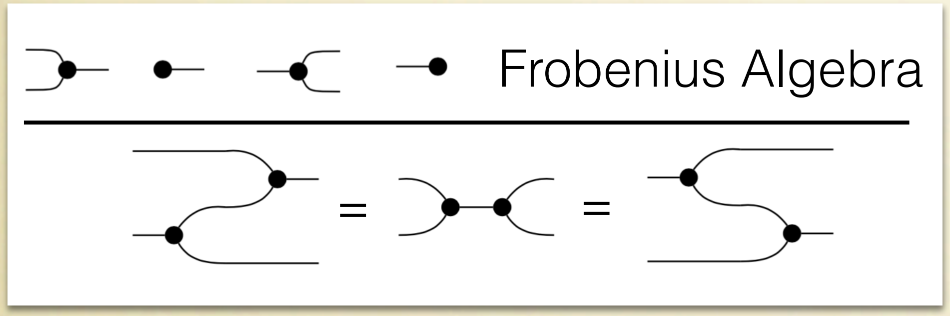
SEVIER Electronic Notes in Theoretical Computer Science 162 (2006) 37–41 www.elsevier.com/locate/e

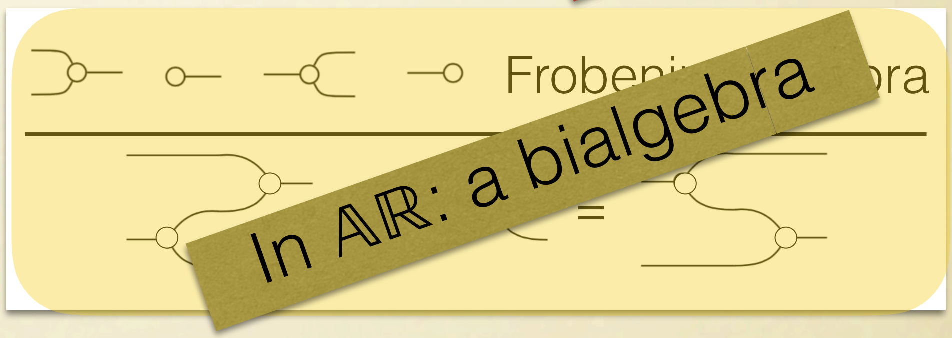
What are the Fundamental Structures of Concurrency? We still don't know!

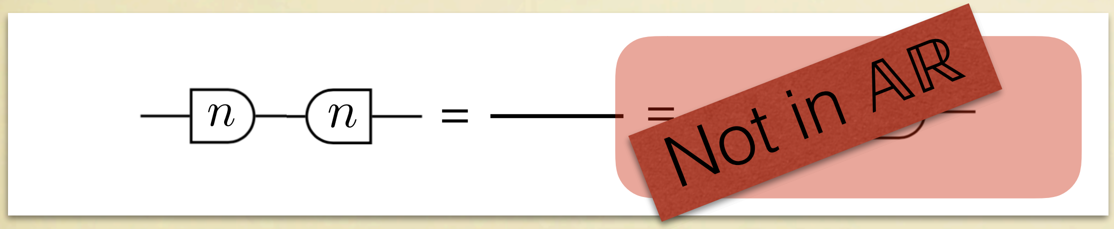
Samson Abramsky [1,2]

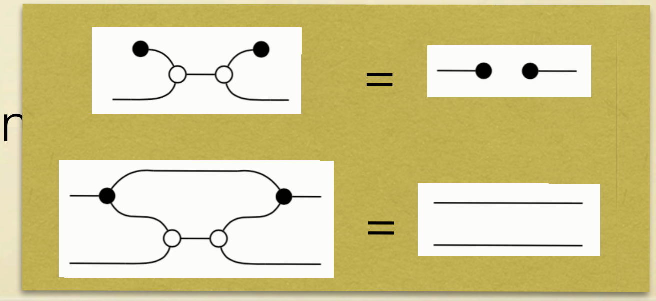


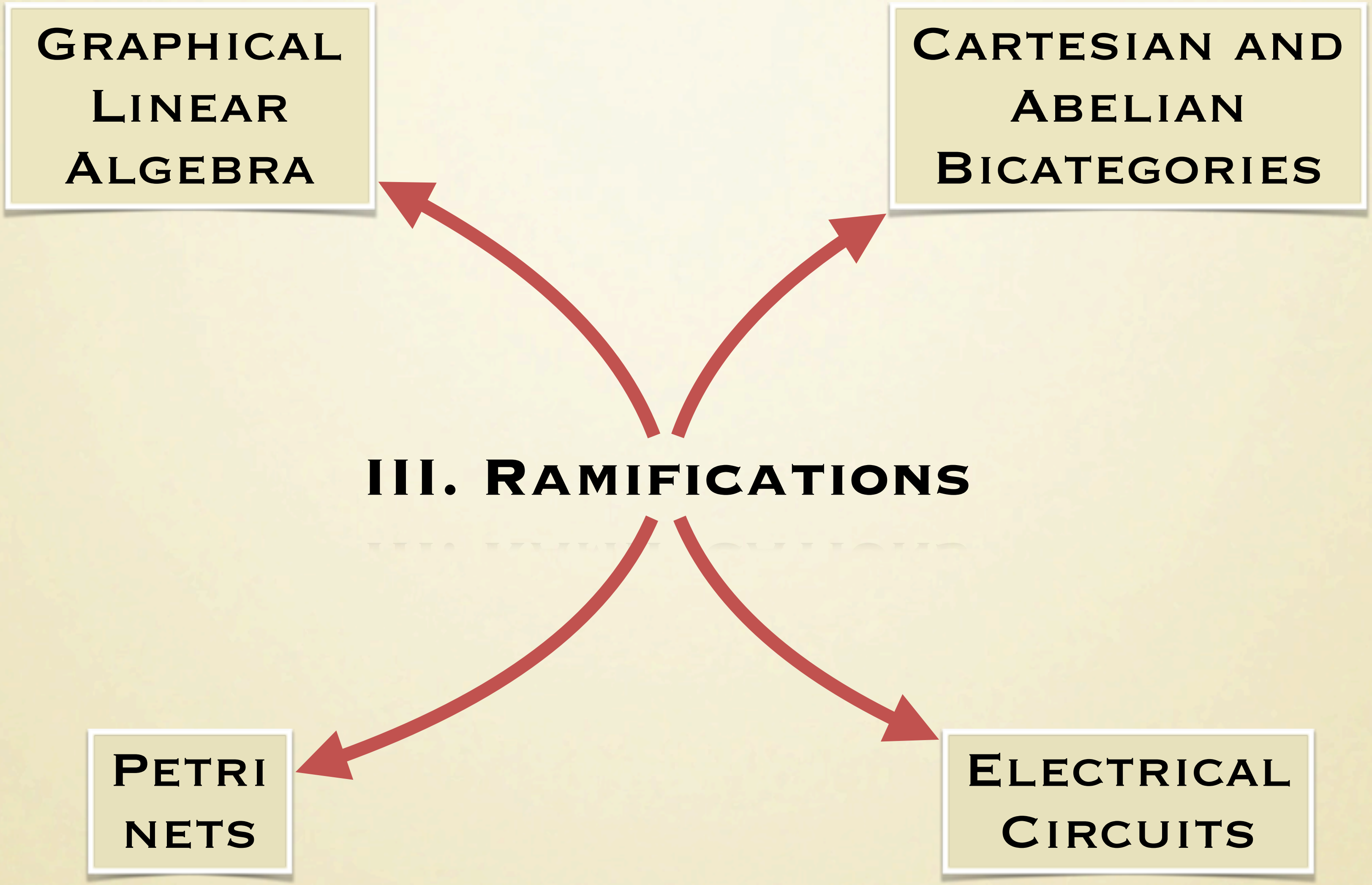






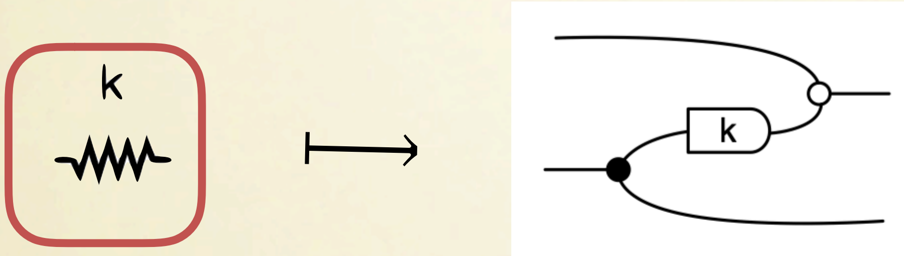




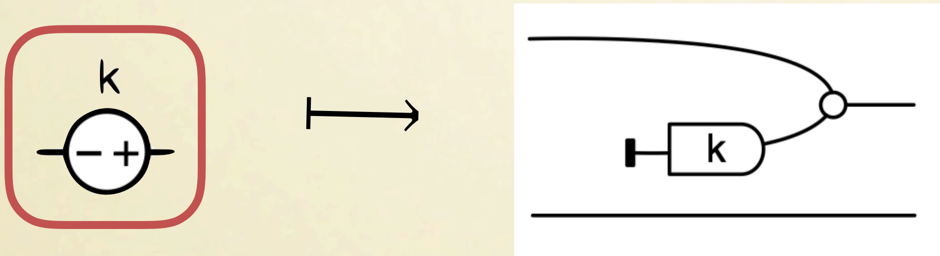

Graphical Linear Algebra
Cartesian and Abelian Bicategories
III. Ramifications
Petri nets
Electrical Circuits

Non Passive Electrical Circuits

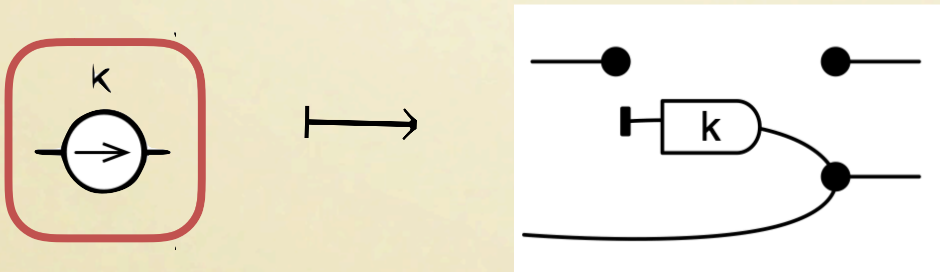
Encode electrical circuits as circuit diagrams



Resistor



Voltage Source

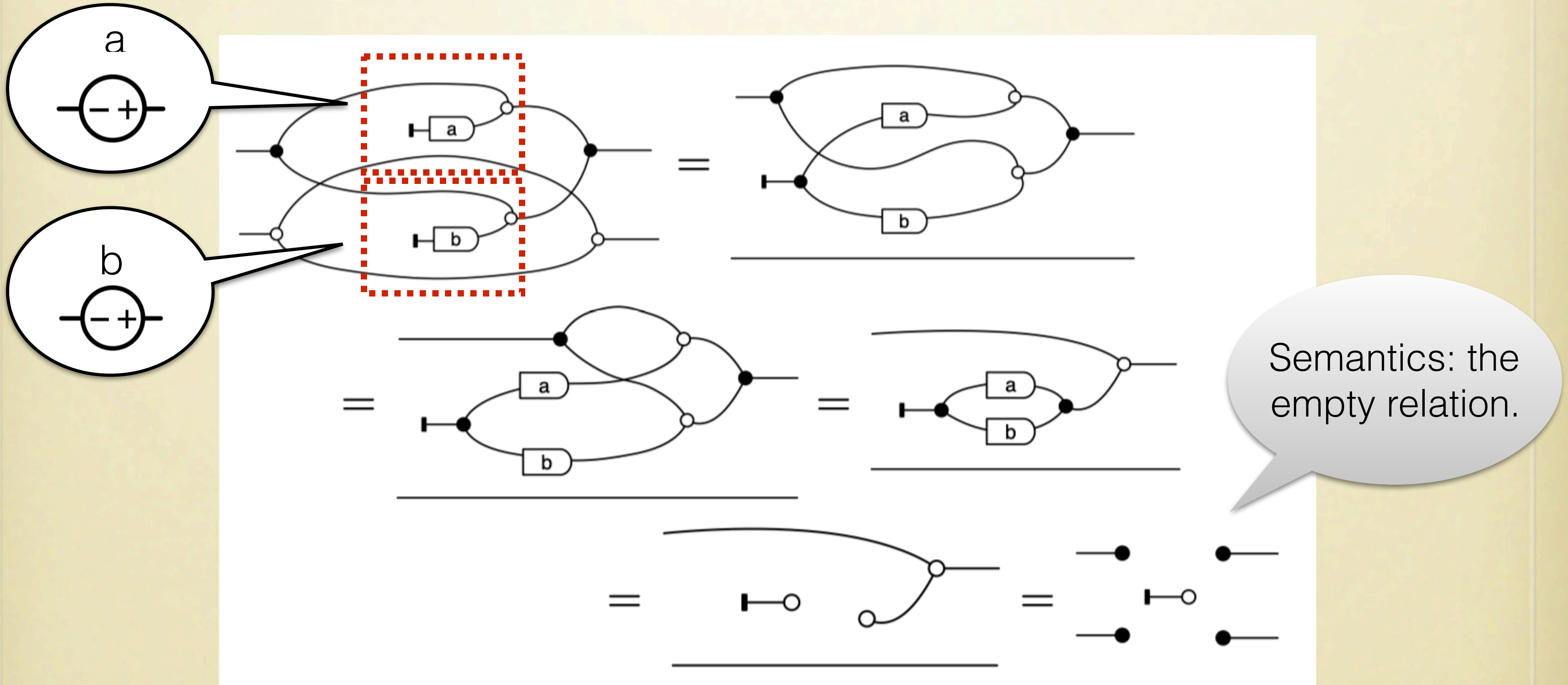


Current Source

Bonchi, Piedeleu, Sobocinski, Zanasi - *Graphical Affine Algebra*, LICS 2019.

Equational Reasoning with Electrical Circuits

Proof that parallel voltage sources of different voltages are disallowed.



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Mahalo