# The Logical Essentials of Bayesian Reasoning

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University College London

Joint work with Bart Jacobs (Radboud University Nijmegen)

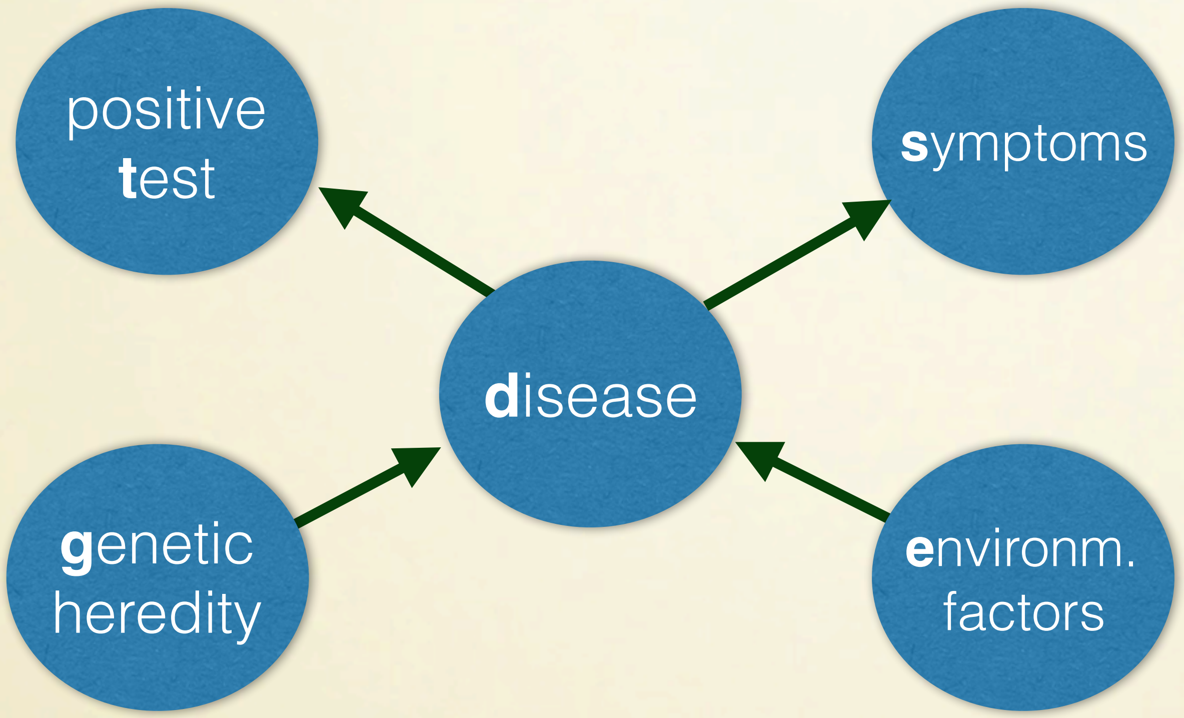
# In a nutshell

- We develop a categorical approach to Bayesian probability theory.

- Our methodology is driven by programming language semantics.

- It offers a principled, compositional way of performing the fundamental Bayesian reasoning tasks, such as inference and learning.

# Bayesian Networks

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| g $e^{\perp}$ | 8/10 |
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## Inference questions

P(t)

*What is the a priori probability of a positive test?*

P(t | s)

*What is the probability of a positive test* ***given*** *the symptoms?*

# Toolbox

| | | | |
|---|---|---|---|
| **State** | $\omega \in \mathcal{D}(X)$ | State of affairs | $1 \multimap X$ |
| **Predicate** | $p : X \to [0,1]$ | Observation, (fuzzy) event | $X \multimap 2$ |
| **Conditioning** | $\omega\|_p \in \mathcal{D}(X)$ | Revision due to an observation | $1 \multimap X$ |
| **Channel** | $f : X \to \mathcal{D}(Y)$ | Change of base/ message passing | $X \multimap Y$ |

State transformer $\omega \in \mathcal{D}(X) \quad \mapsto \quad (f \gg \omega) \in \mathcal{D}(Y)$

Predicate transformer $q \colon Y \to [0,1] \quad \mapsto \quad (f \ll q) \colon X \to [0,1]$

Type as arrows of kl($\mathcal{D}$)

# Toolbox

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State trans

Predicate tra

e as arrows of $\mathrm{kl}(\mathcal{D})$

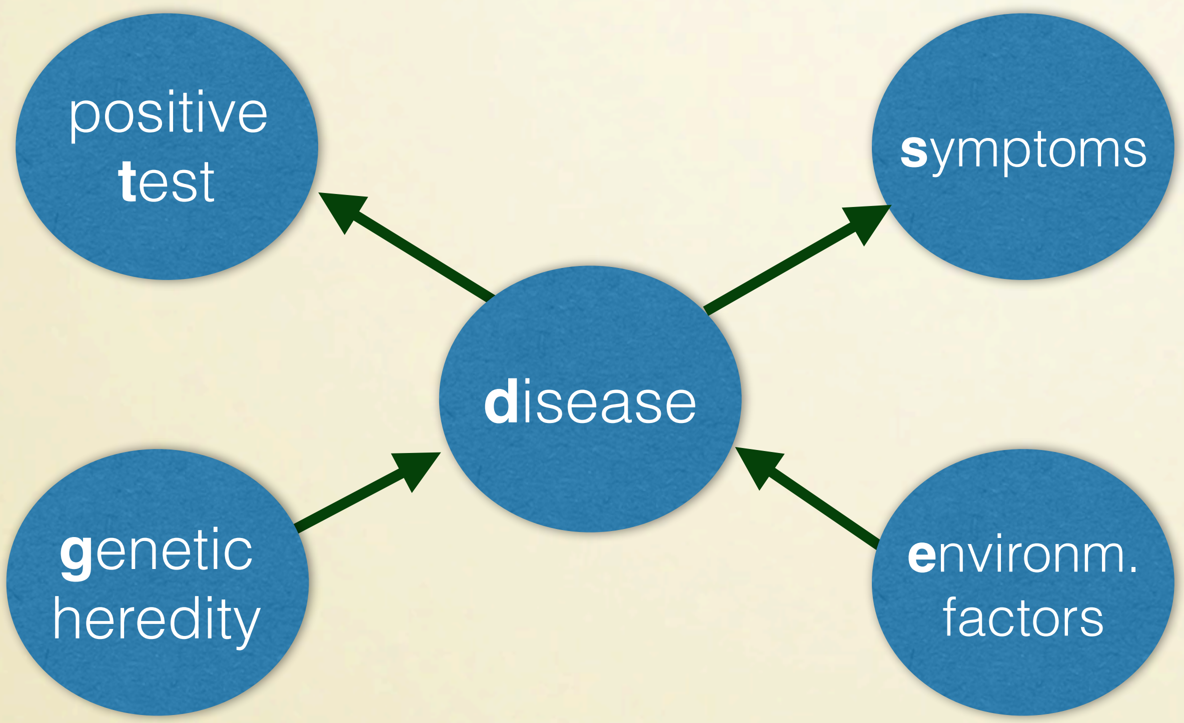
These notions make sense in other categories as well.

Continuous case: $\mathrm{kl}(\mathcal{G})$
states are probability measures, predicates are measurable functions to [0,1].

Quantum case: $\mathbb{vNA}^{op}$
states are…quantum states, predicates are effects.

# Bayesian Networks in Kleisli

We interpret a Bayesian network as an arrow of **kl**($\mathcal{D}$).

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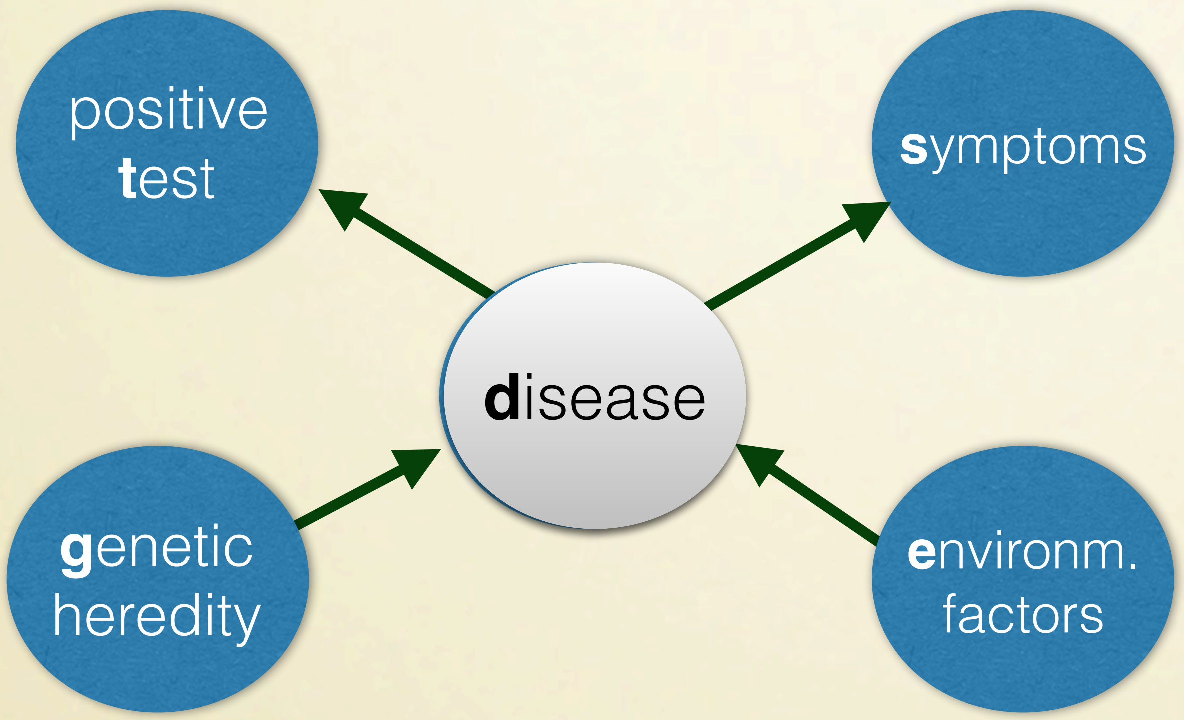


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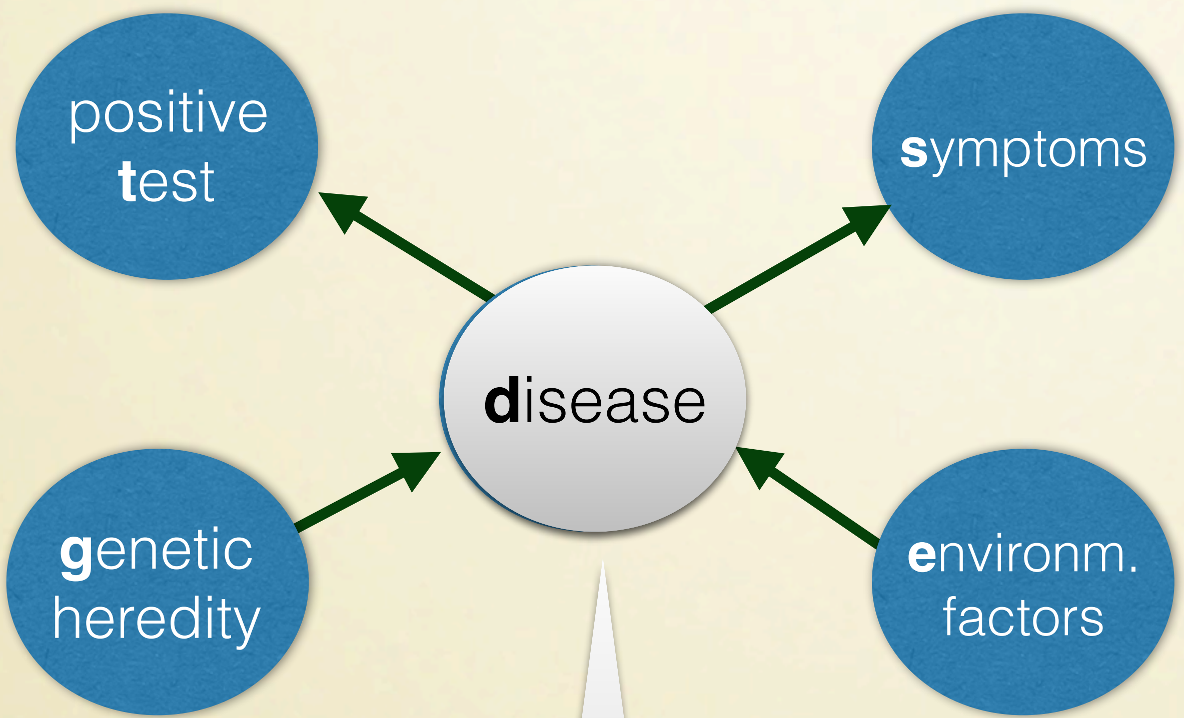


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$$\{g, g^{\perp}\} \times \{e, e^{\perp}\} \overset{\mathrm{D}}{\rightsquigarrow} \{d, d^{\perp}\}$$

$\mathrm{Kl}(\mathcal{D})$

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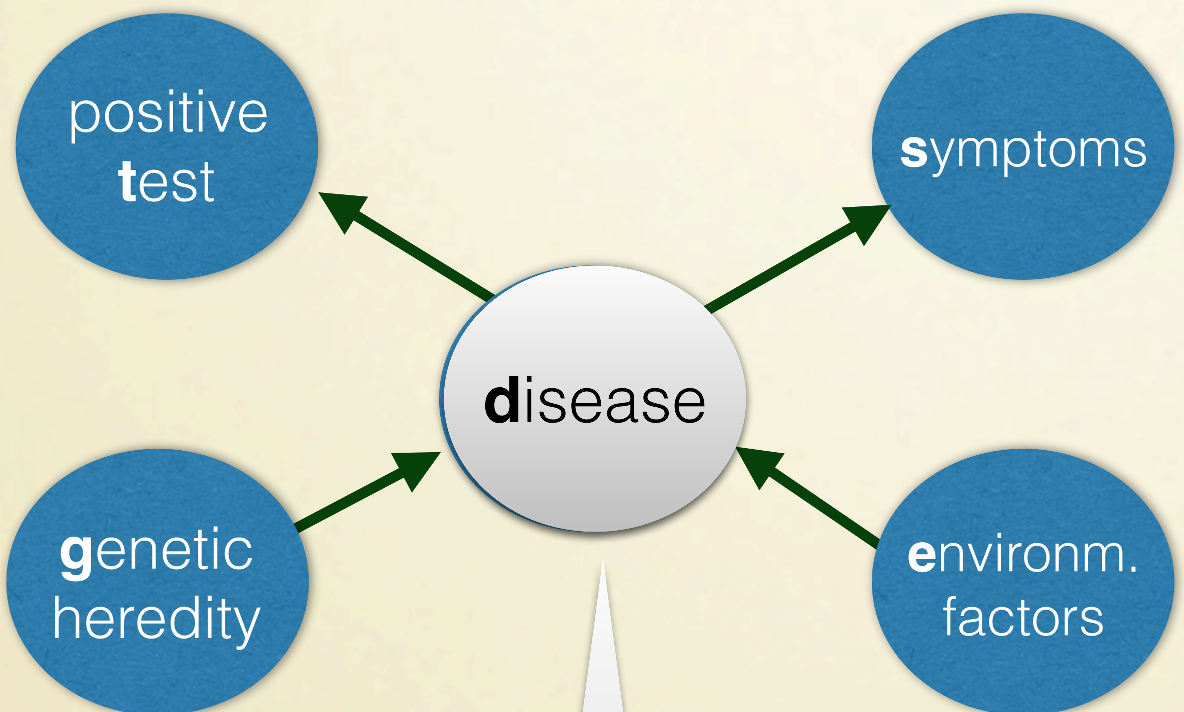


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Kl($\mathcal{D}$)

$$\{g, g^\perp\} \times \{e, e^\perp\} \overset{D}{\multimap} \{d, d^\perp\}$$

$$\begin{aligned}(g, e) &\mapsto 9/10|d\rangle + 1/10|d^\perp\rangle \\ (g, e^\perp) &\mapsto 8/10|d\rangle + 2/10|d^\perp\rangle \\ (g^\perp, e) &\mapsto 4/10|d\rangle + 6/10|d^\perp\rangle \\ (g^\perp, e^\perp) &\mapsto 1|d^\perp\rangle\end{aligned}$$

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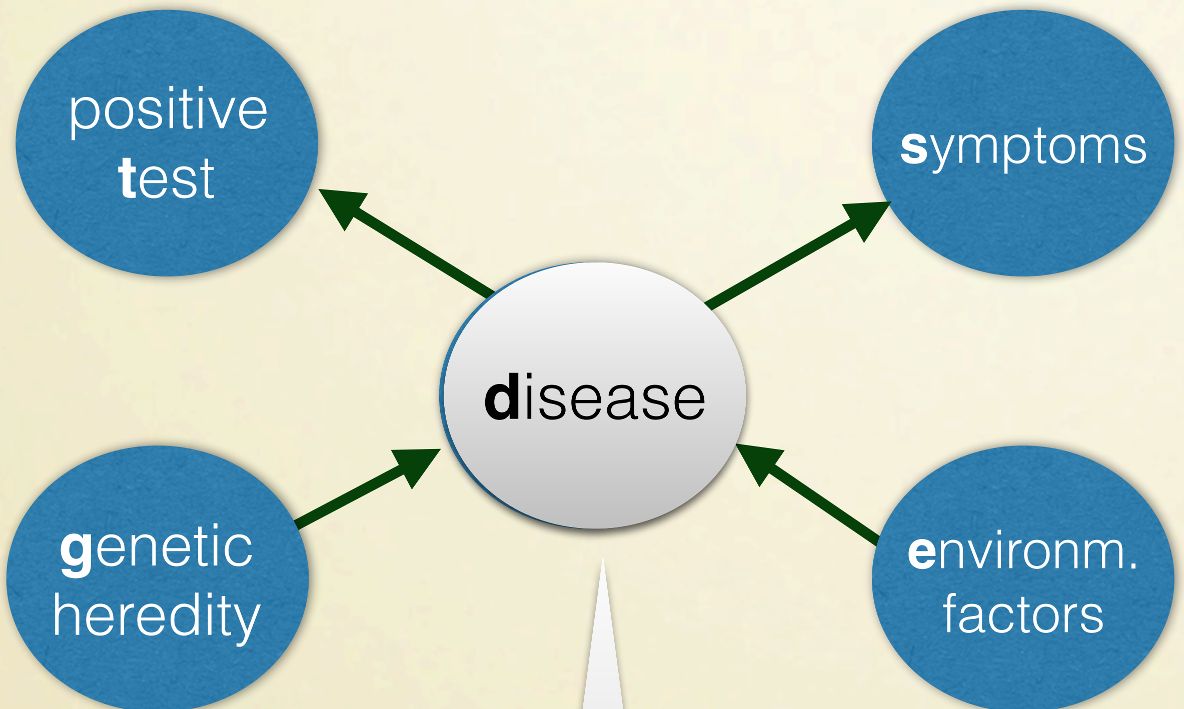


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Kl($\mathcal{D}$)

$$2_g \otimes 2_e \overset{D}{\multimap} 2_d$$

$$(g, e) \mapsto 9/10|d\rangle + 1/10|d^\perp\rangle$$
$$(g, e^\perp) \mapsto 8/10|d\rangle + 2/10|d^\perp\rangle$$
$$(g^\perp, e) \mapsto 4/10|d\rangle + 6/10|d^\perp\rangle$$
$$(g^\perp, e^\perp) \mapsto 1|d^\perp\rangle$$

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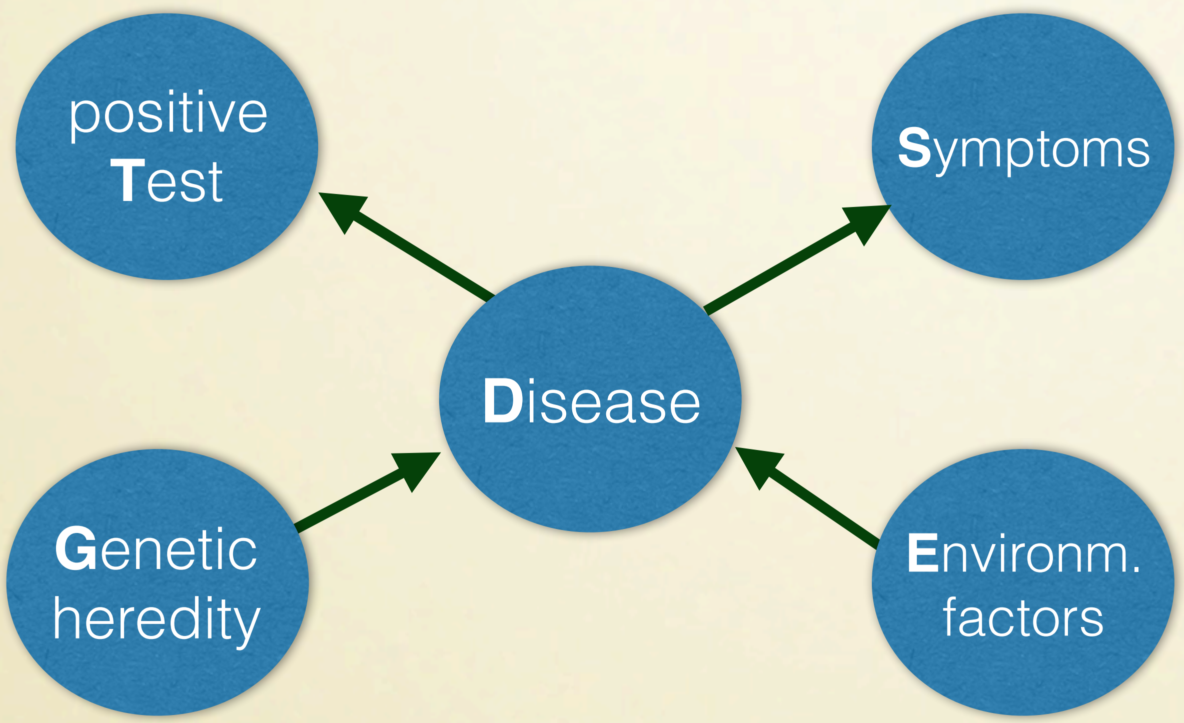


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Kl($\mathcal{D}$)

$$1 \xrightarrow{G\otimes E} 2_G \otimes 2_E \xrightarrow{D} 2_D \xrightarrow{\Delta} 2_D \otimes 2_D \xrightarrow{T\otimes S} 2_T \otimes 2_S$$

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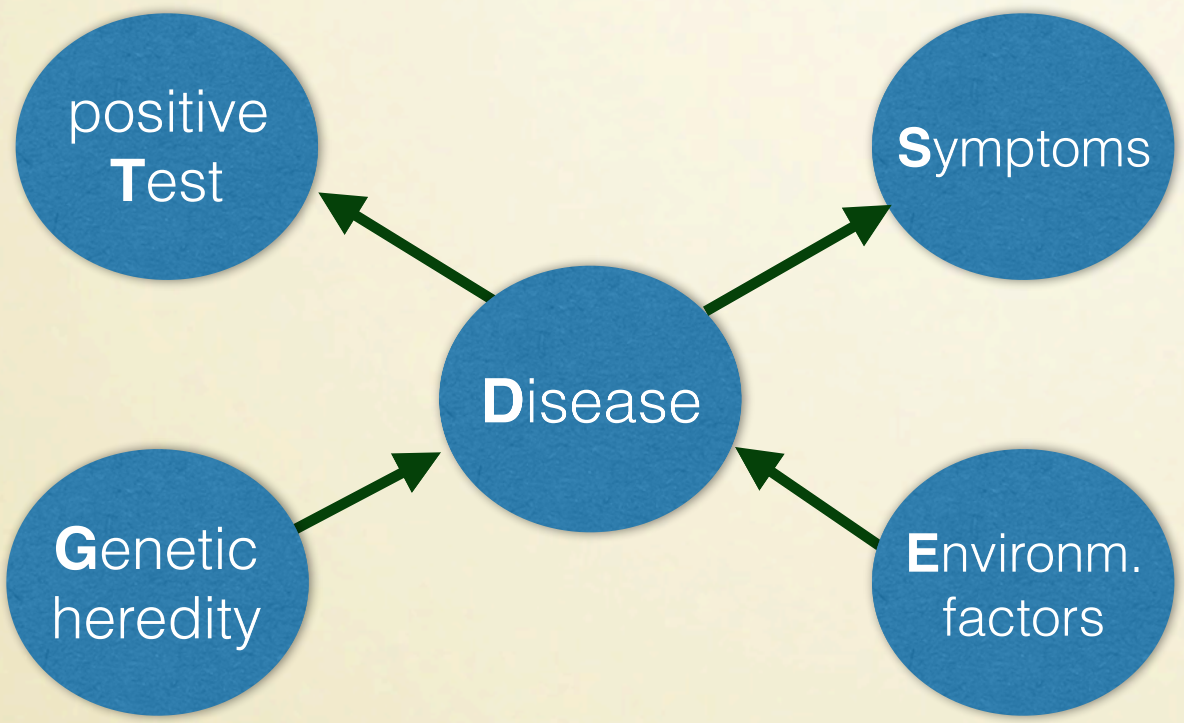


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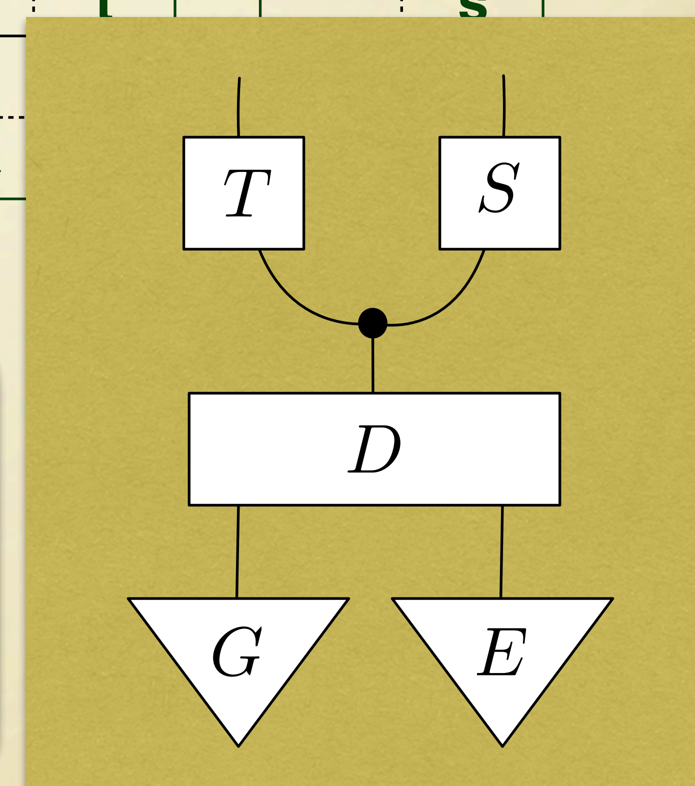



Kl($\mathcal{D}$)

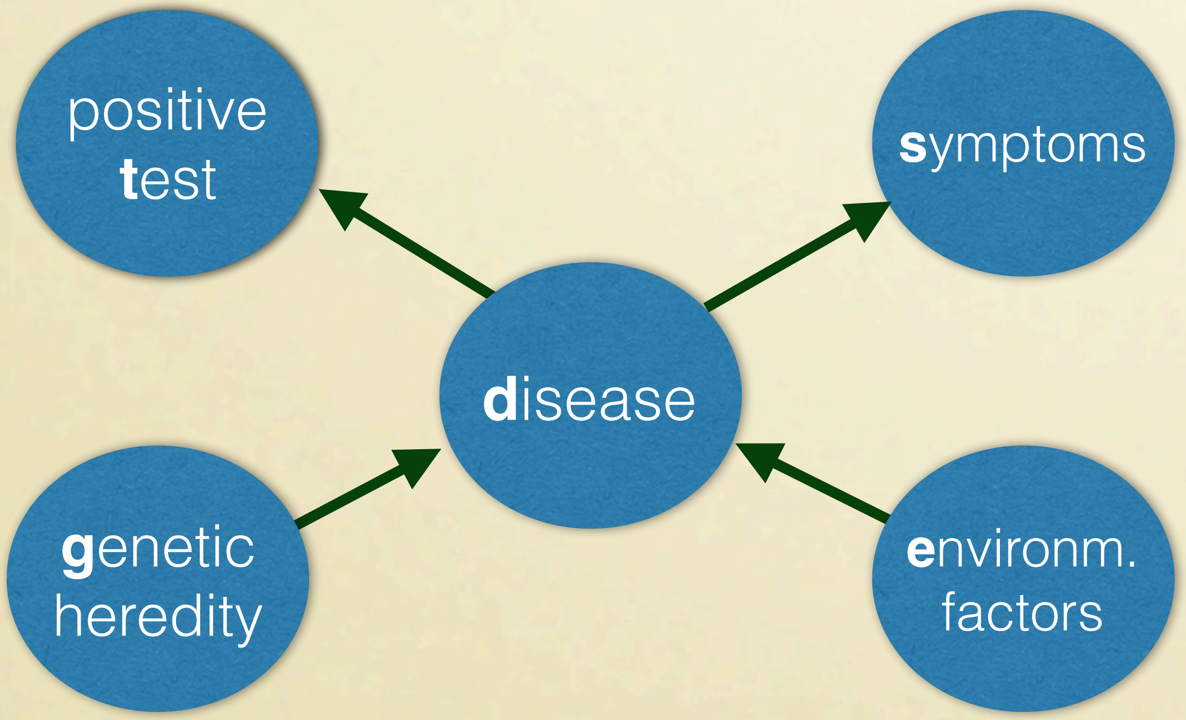
$$1 \xrightarrow{G\otimes E} 2_G \otimes 2_E \xrightarrow{D} 2_D \xrightarrow{\Delta} 2_D \otimes 2_D \xrightarrow{T\otimes S} 2_T \otimes 2_S$$

# Bayesian Inference in Kleisli

Both Bayesian networks and our toolbox live in **kl**($\mathcal{D}$).

We shall now compute the two inference questions in **kl**($\mathcal{D}$).

The calculation will have a \`dynamical' flavour:




**Inference questions**

*What is the a priori probability of a positive test?*

*What is the probability of a positive test* ***given*** *the symptoms?*

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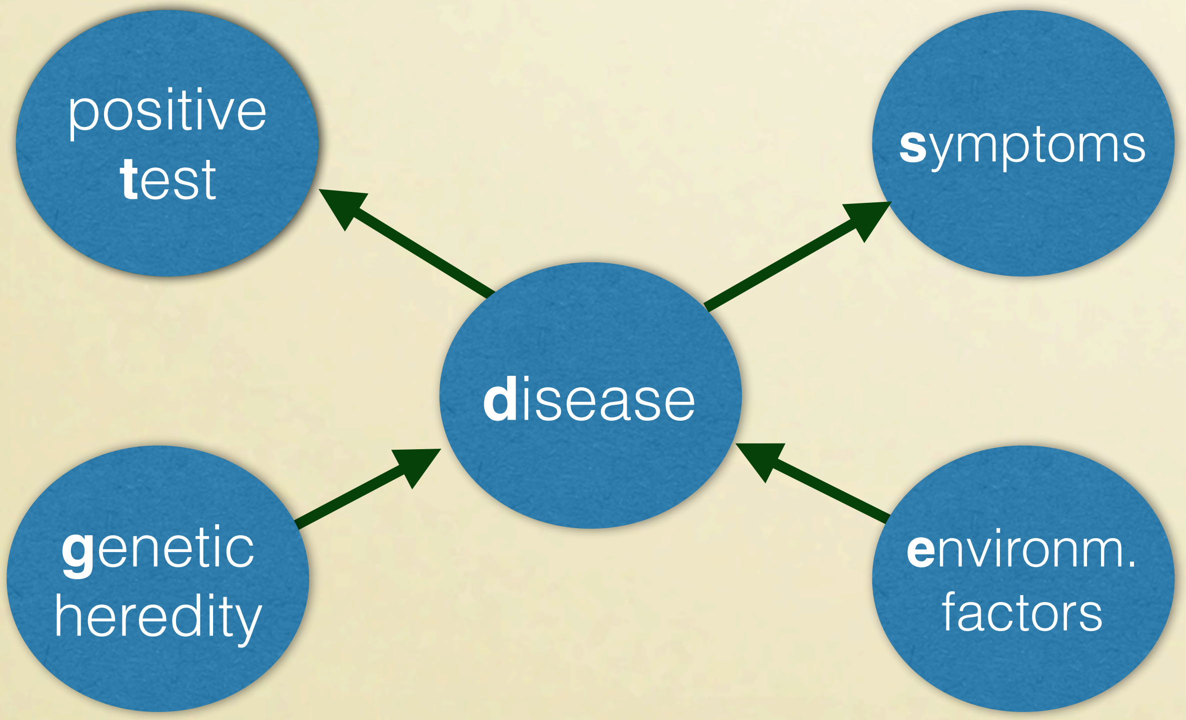


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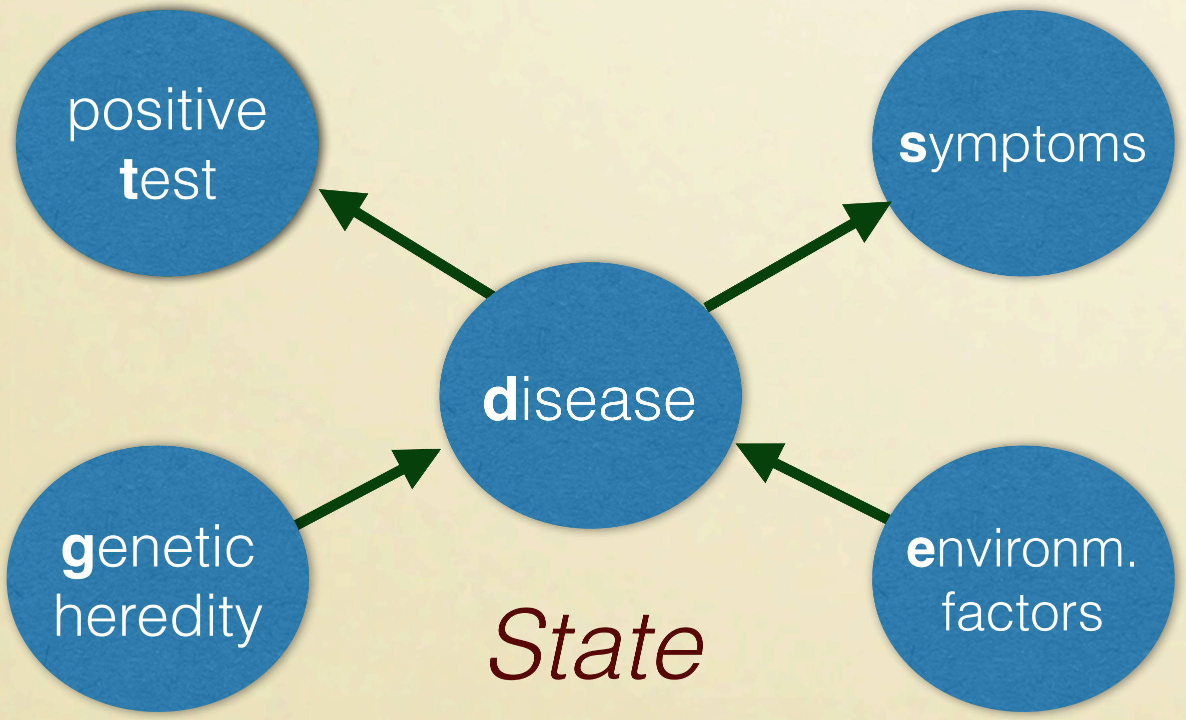


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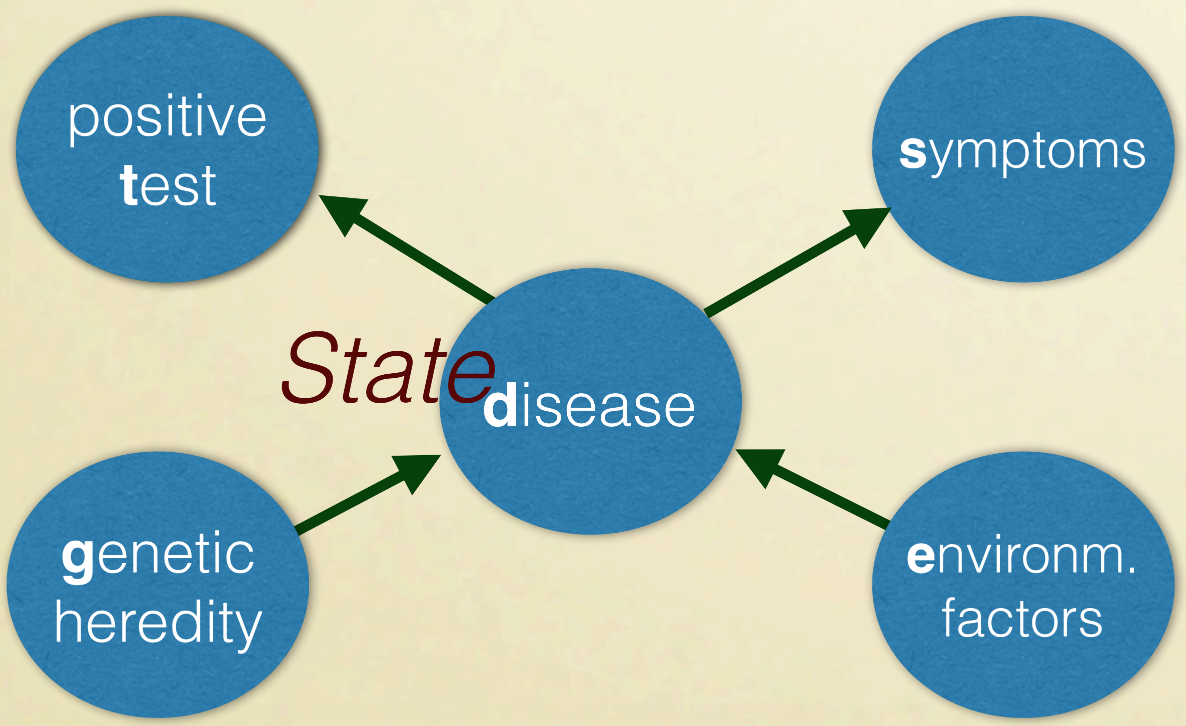


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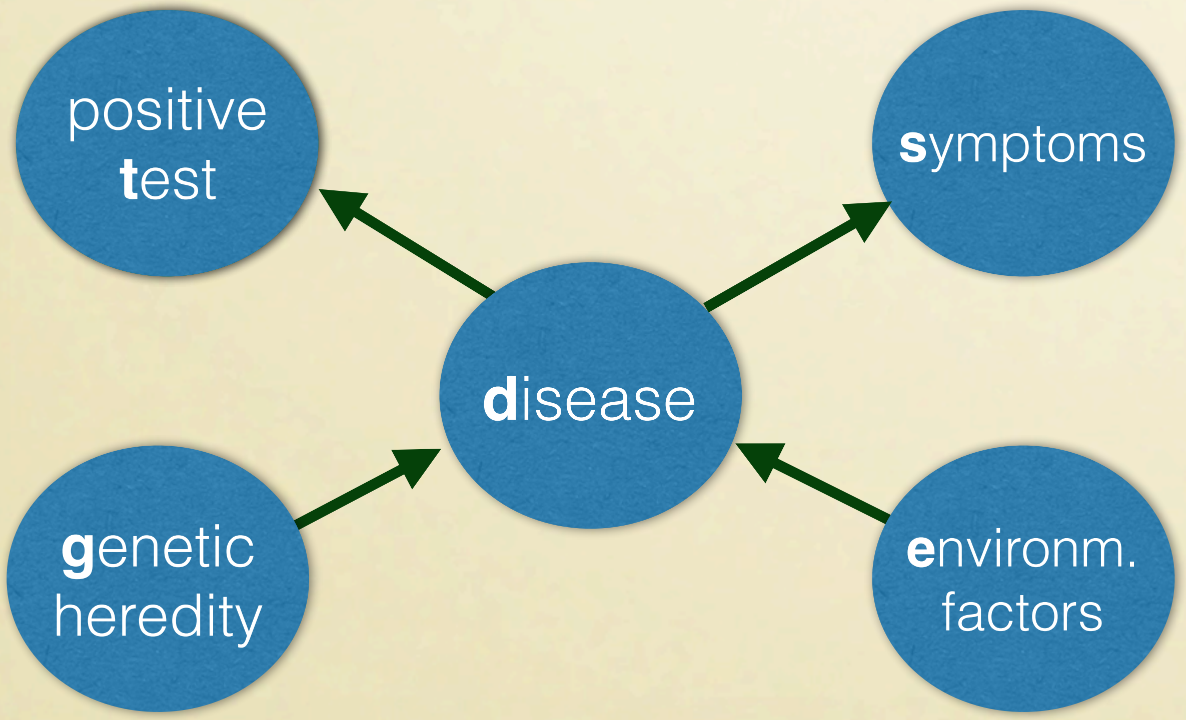
*State*

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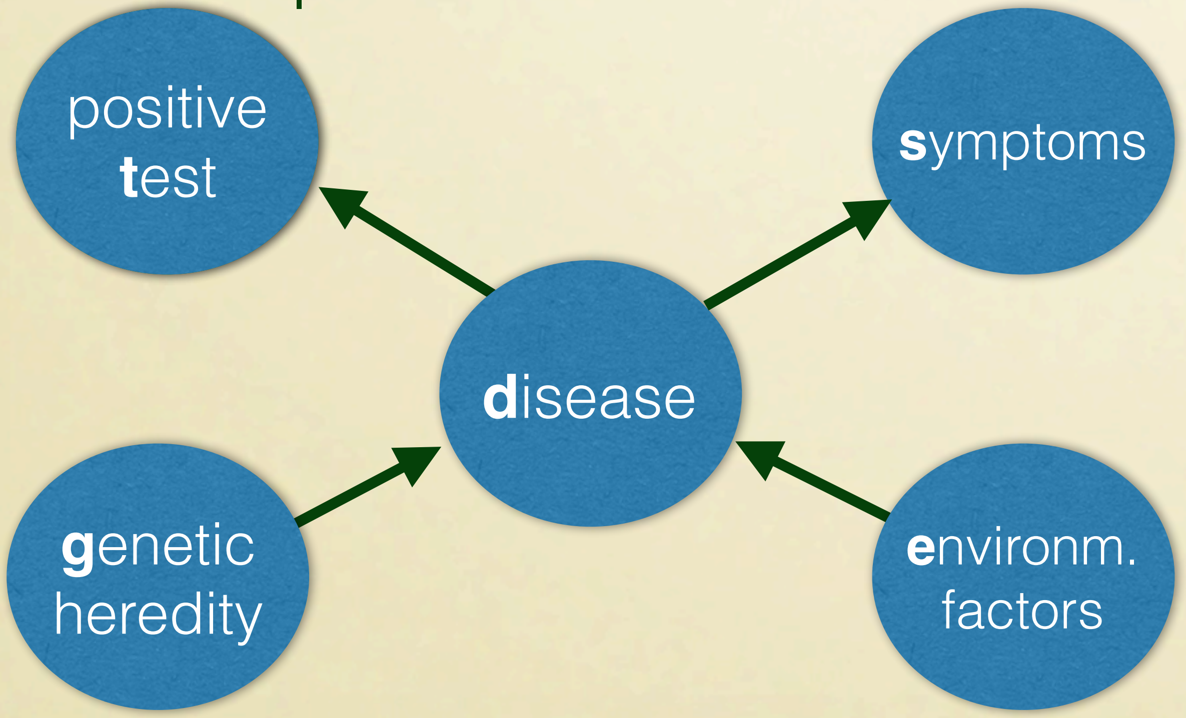
$State|_{PosTest}$

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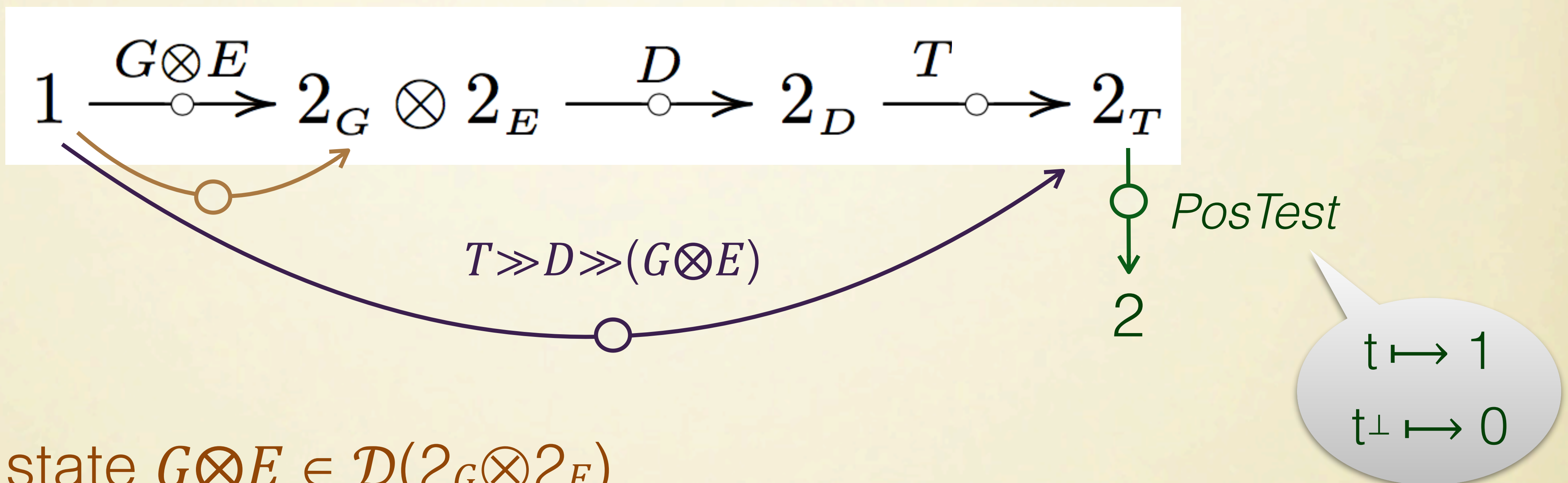
*What is the a priori probability of a positive test?*

# Bayesian Inference in Kleisli

**Inference question I** *What is the a priori probability of a positive test?*




1. Consider state $G{\otimes}E \in \mathcal{D}(2_G{\otimes}2_E)$

2. Use channel $T$ and $D$ as state transformers

$$T{\gg}D{\gg}(G{\otimes}E) \in \mathcal{D}(2_T)$$

3. Consider the predicate *PosTest*: $2_T \to [0,1]$

4. Answer is the conditioned state $(T{\gg}D{\gg}(G{\otimes}E))_{PosTest} \in \mathcal{D}(2_T)$

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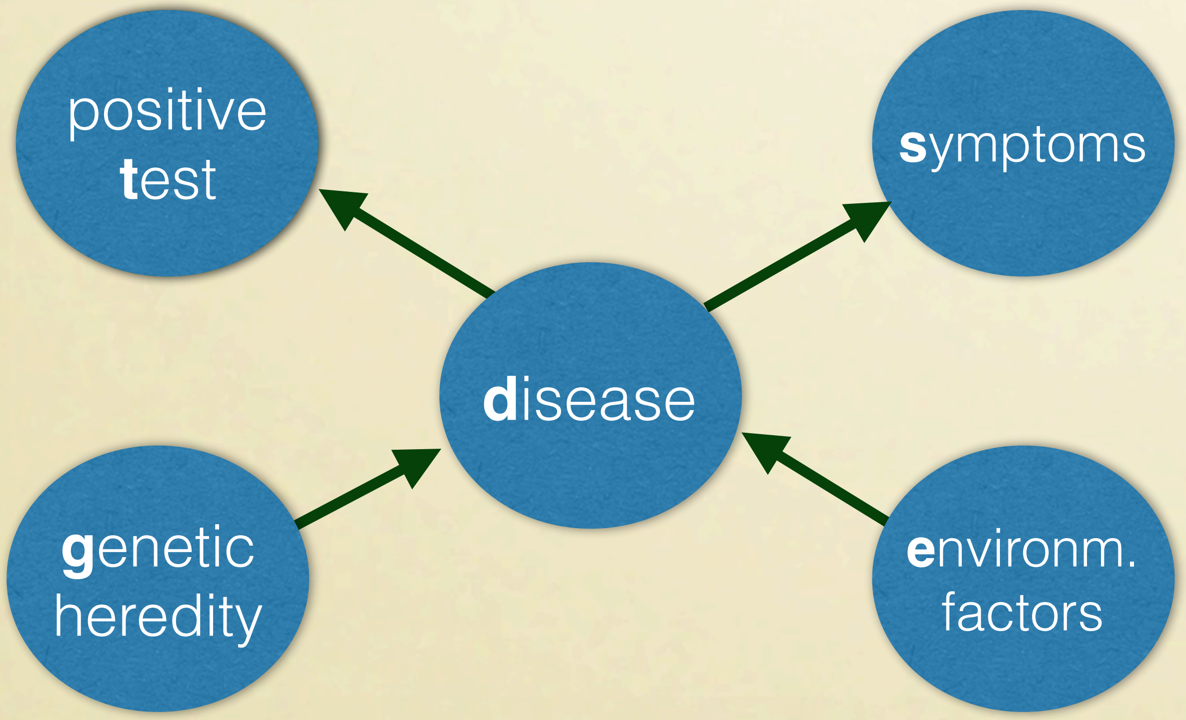


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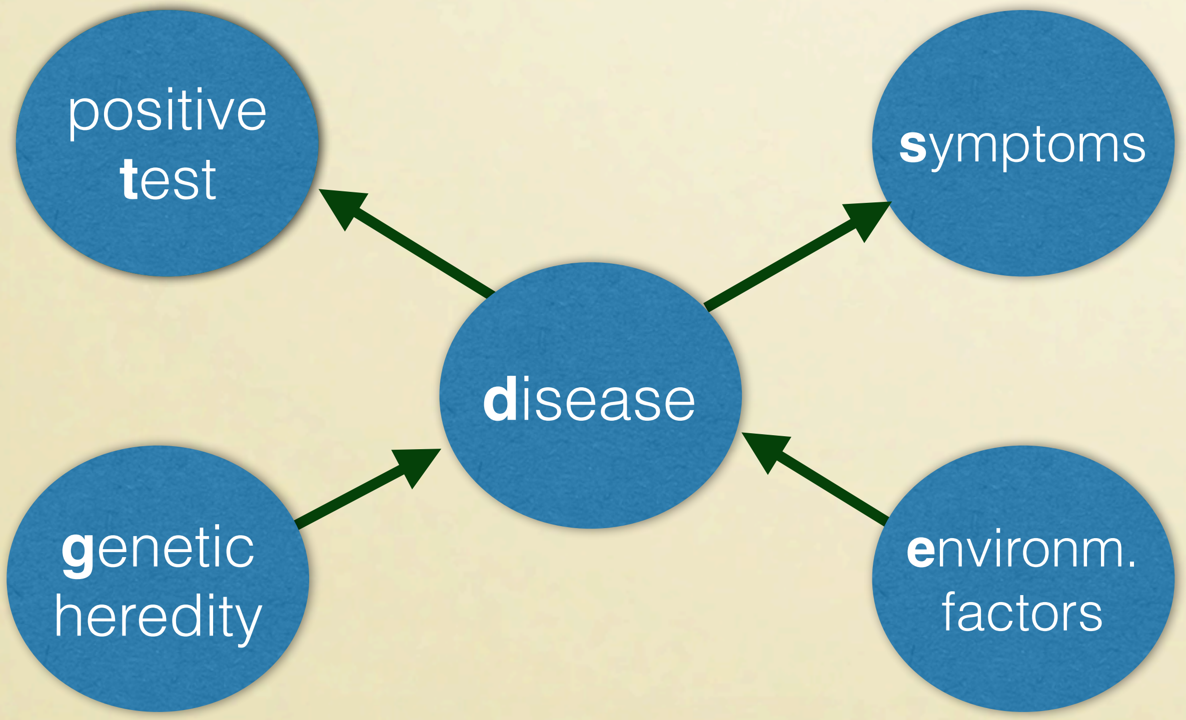


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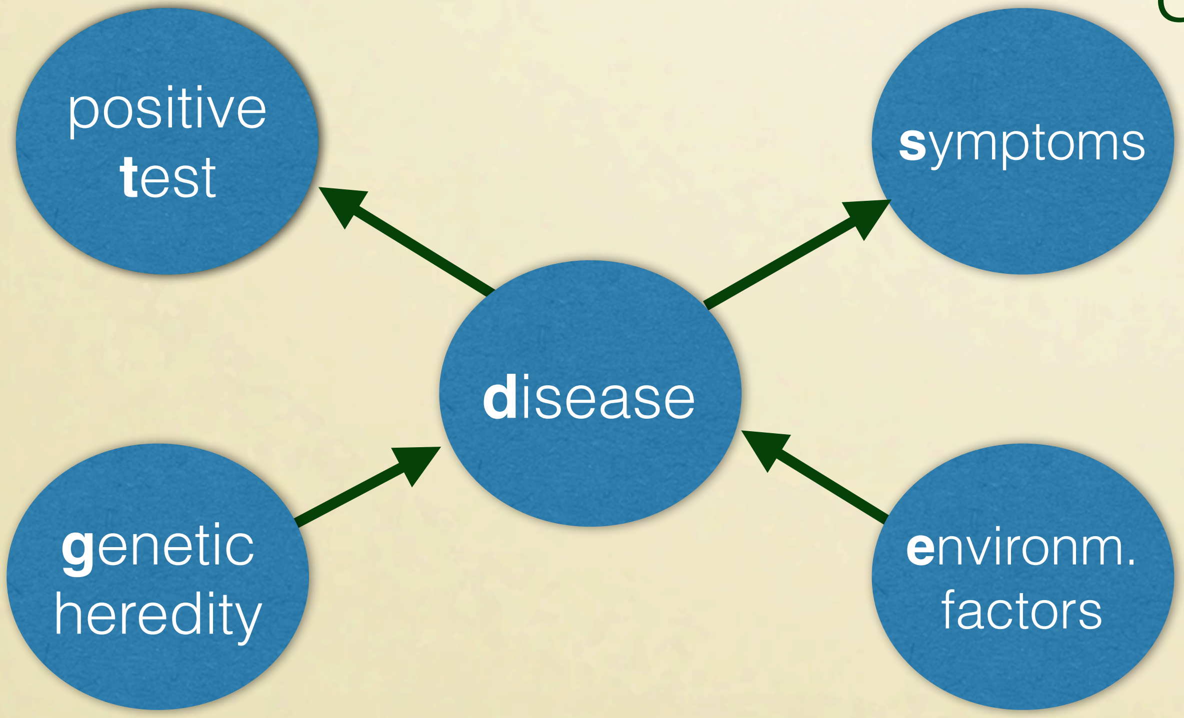
*ObSy*

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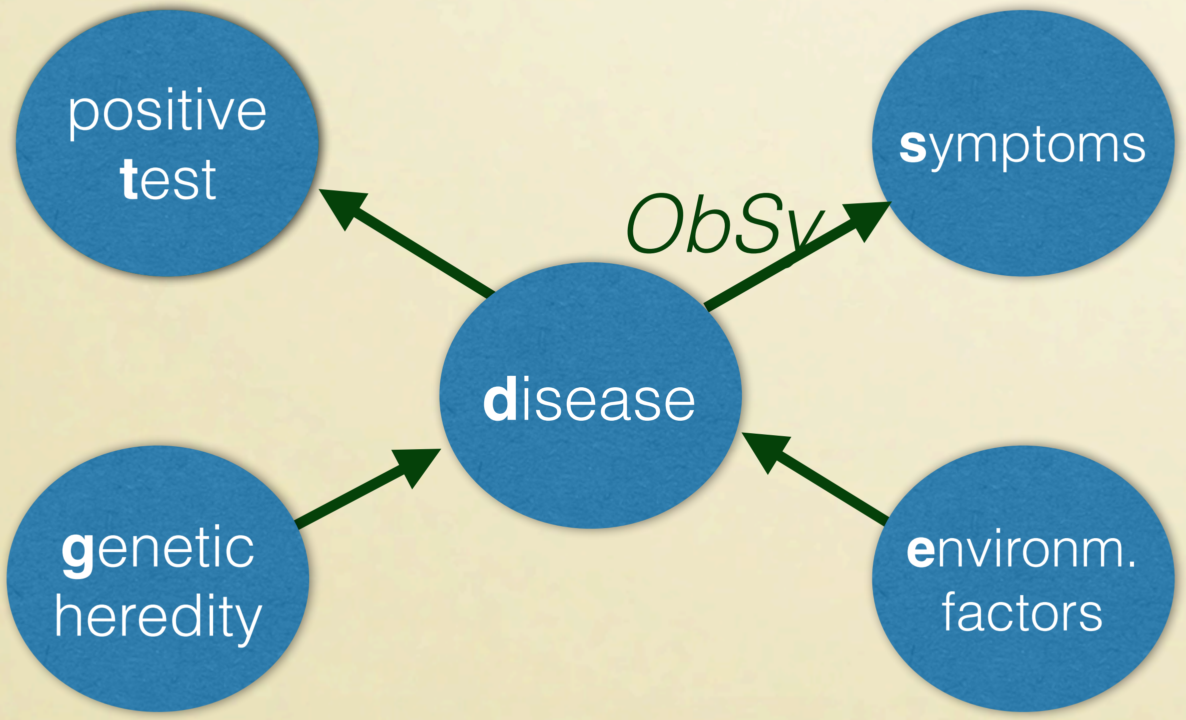


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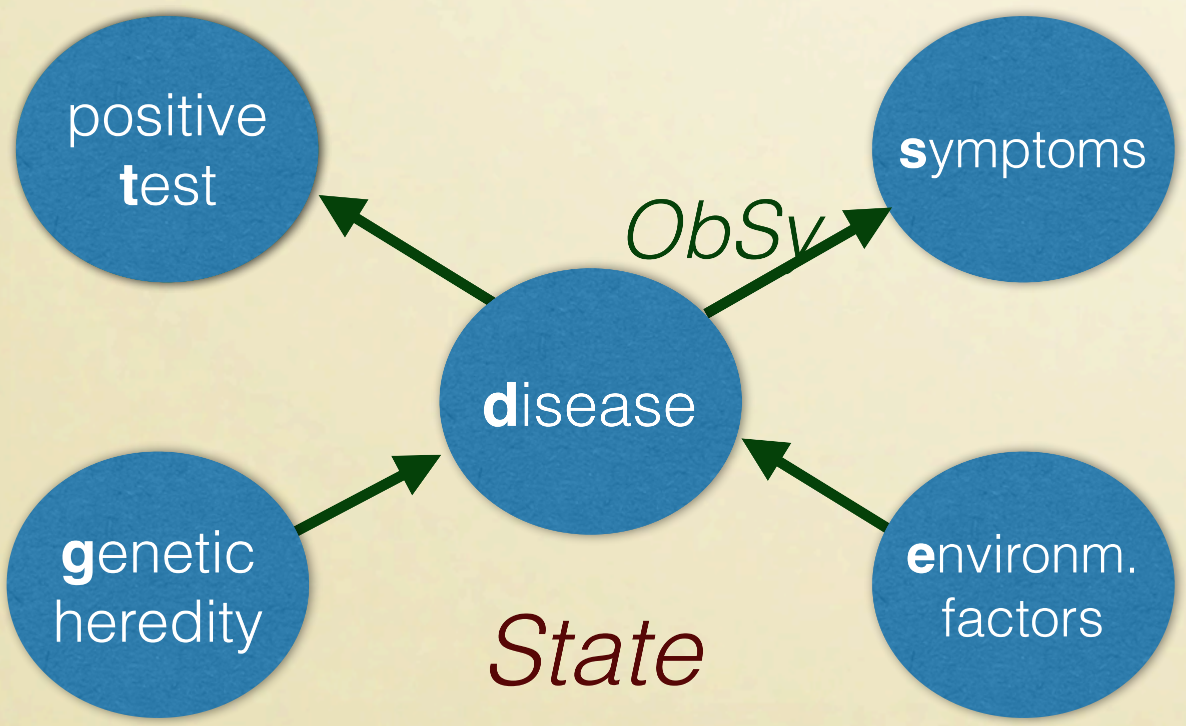


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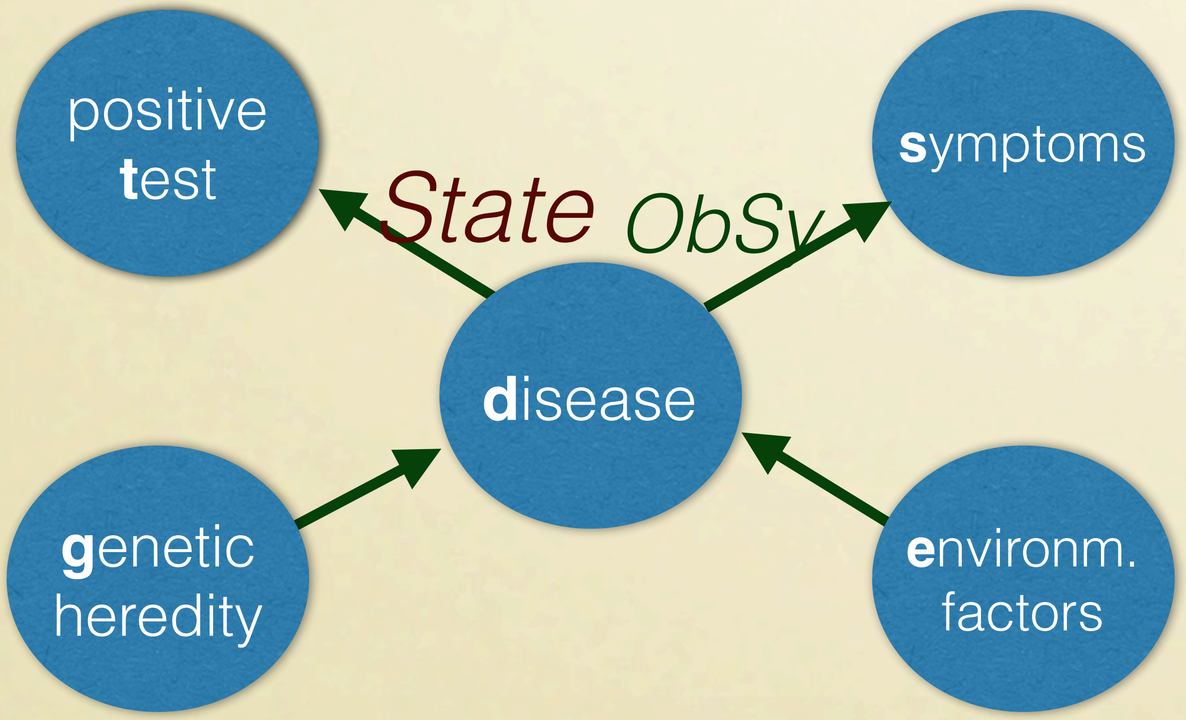


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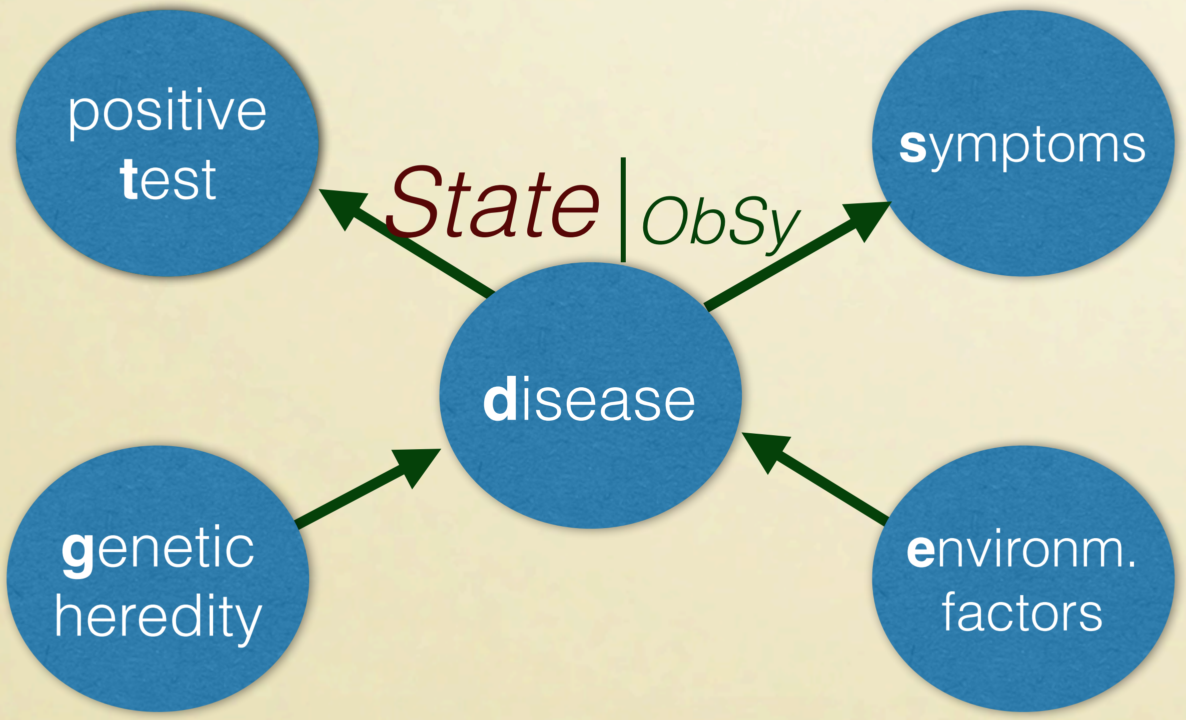


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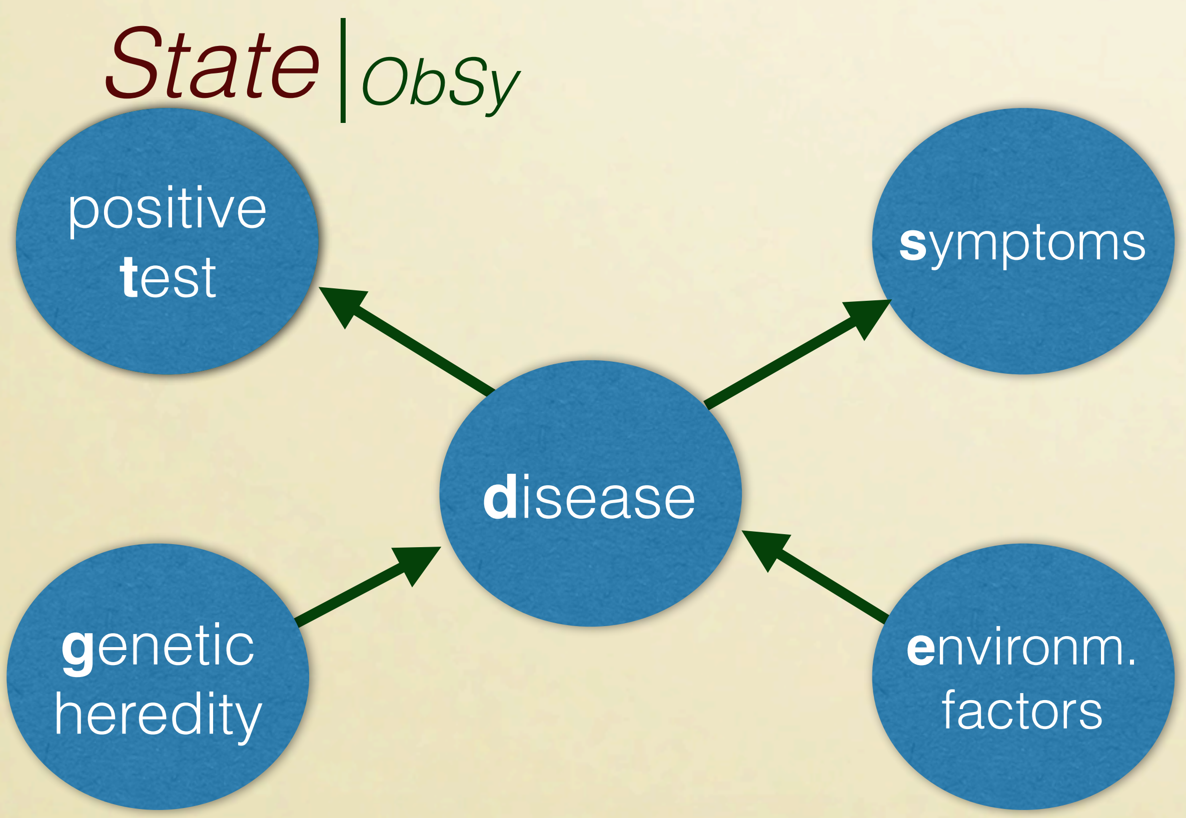


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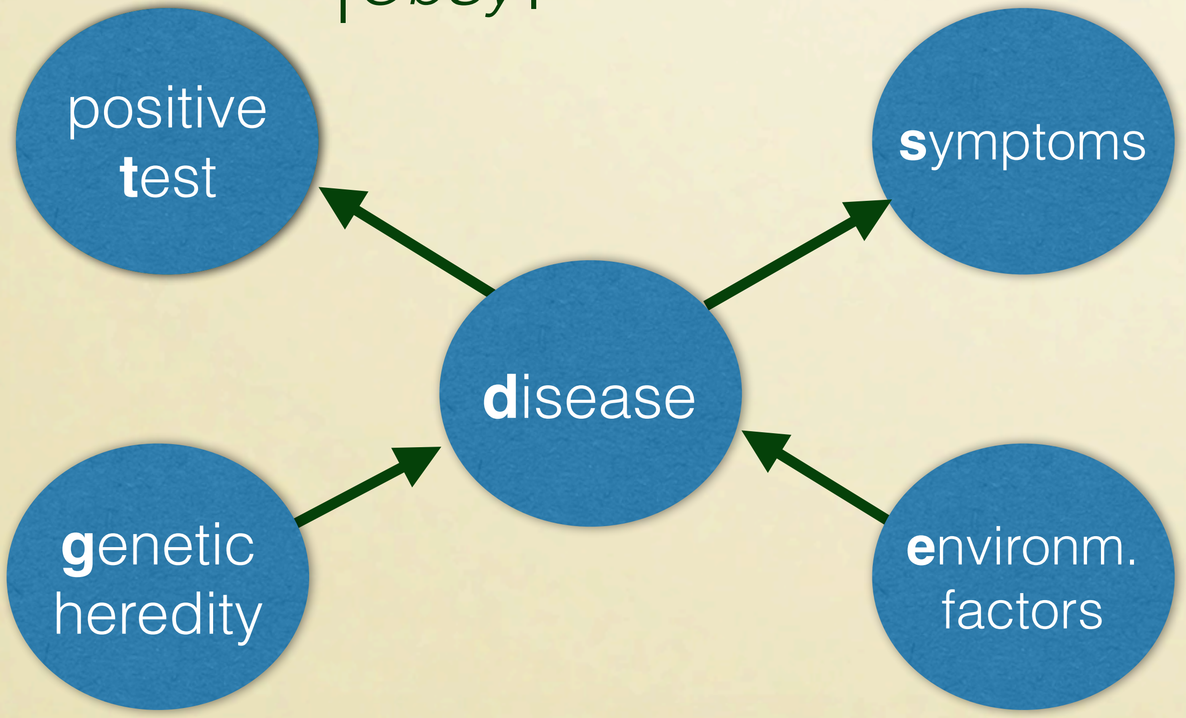
The calculation will have a `dynamical' flavour:

$State|_{ObSy}|_{PosTest}$

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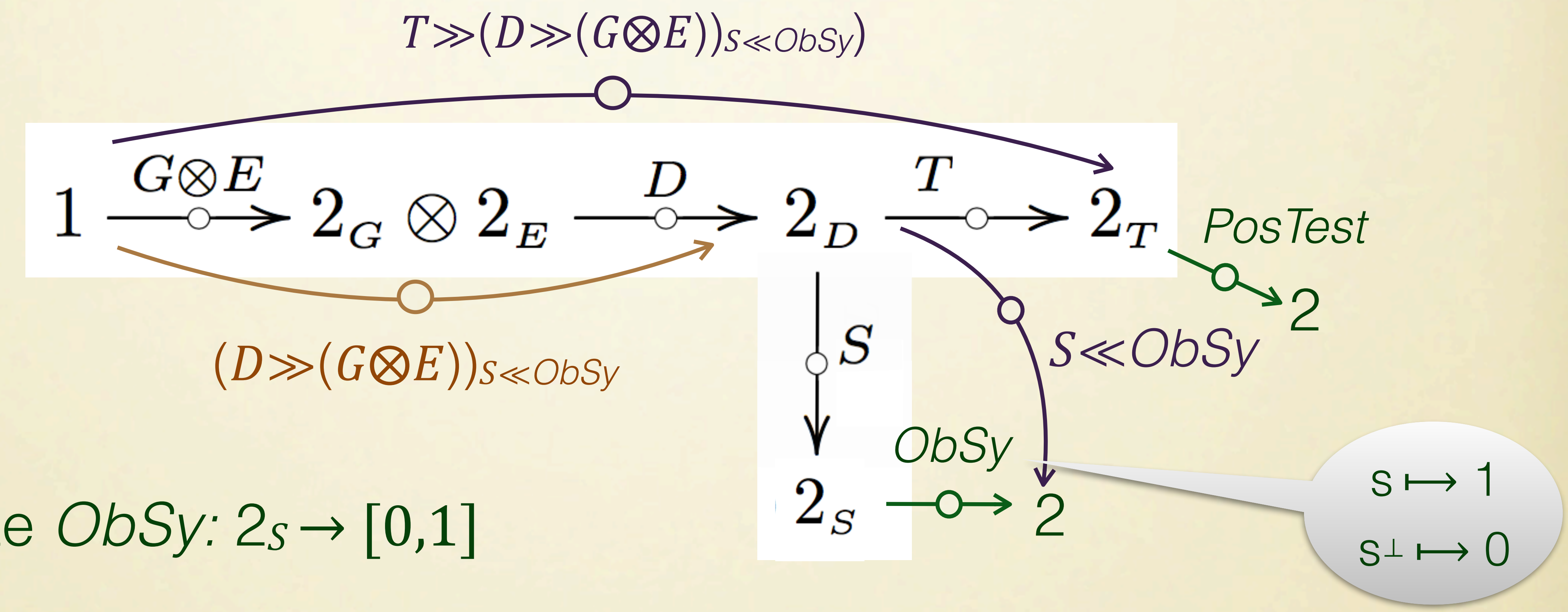
# Bayesian Inference in Kleisli

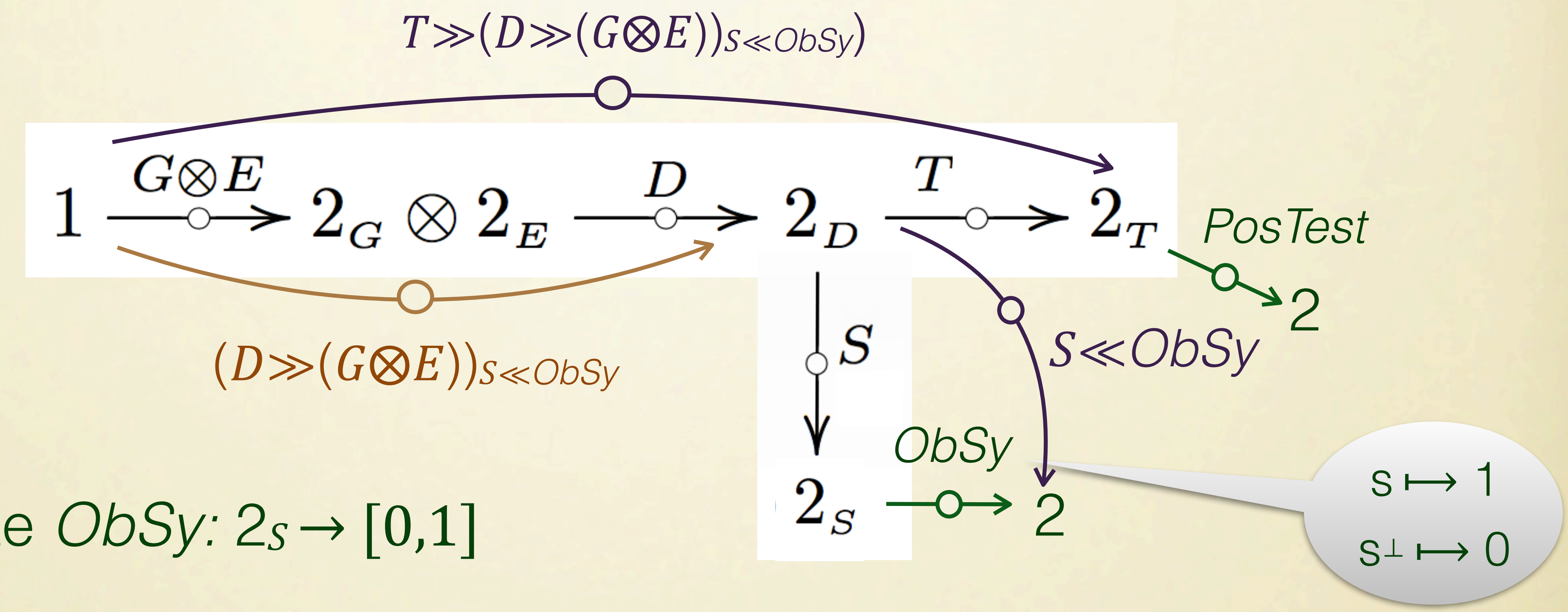
**Inference question II** *What is the probability of a positive test **given** the symptoms?*




1. Predicate *ObSy*: $2_S \to [0,1]$
2. Predicate transformation $S\ll ObSy : 2_D \to [0,1]$
3. Conditioning $(D\gg(G\otimes E))_{S\ll ObSy} \in \mathcal{D}(2_D)$
4. State transformation $T\gg(D\gg(G\otimes E))_{S\ll ObSy}) \in \mathcal{D}(2_T)$
5. Answer is the conditioned state $(T\gg(D\gg(G\otimes E))_{S\ll ObSy}))_{PosTest} \in \mathcal{D}(2_T)$

# Bayesian Inference in Kleisli

**Inference question II** *What is the probability of a positive test **given** the symptoms?*

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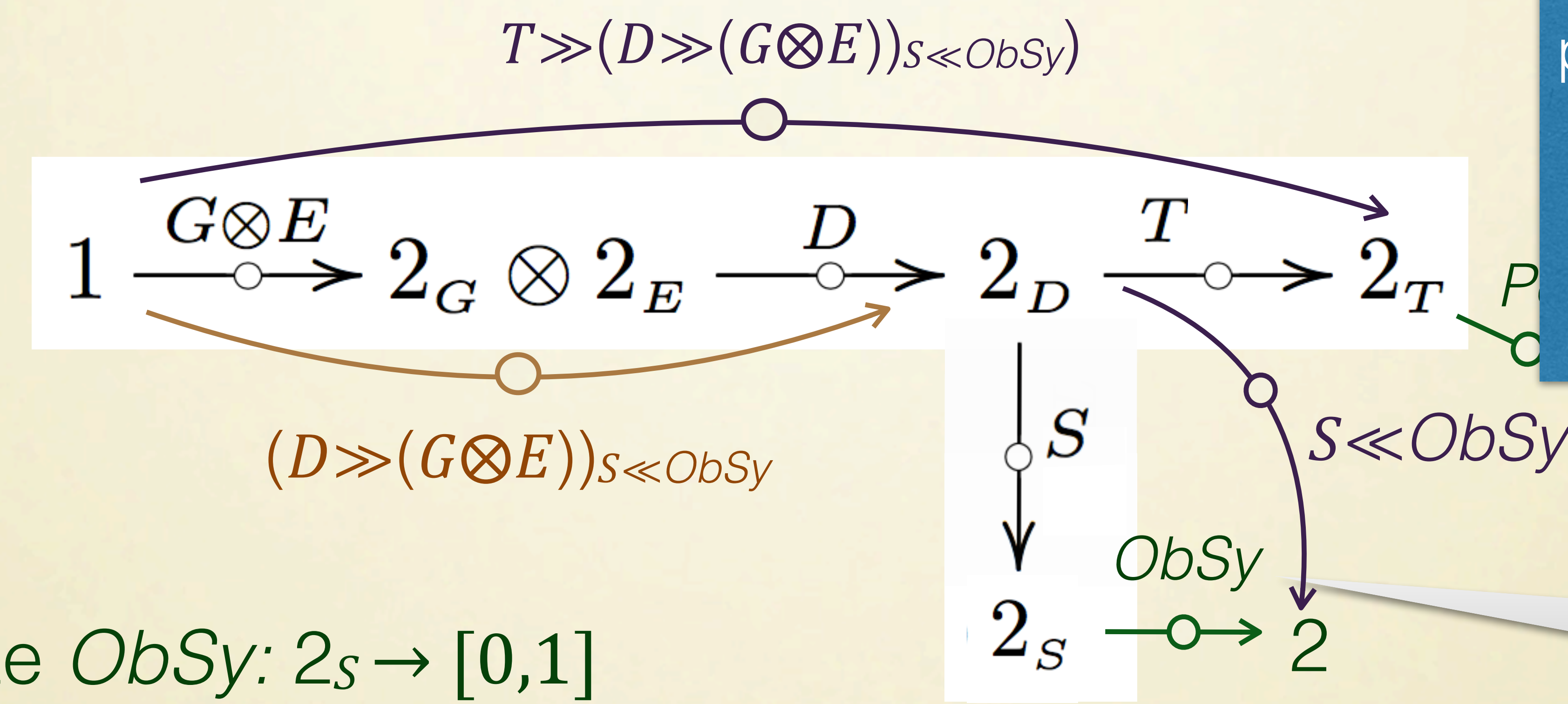
4. State transformation $T \gg (D \gg (G \otimes E))_{S \ll ObSy}) \in \mathcal{D}(2_T)$
5. Answer is the conditioned state $(T \gg (D \gg (G \otimes E))_{S \ll ObSy}))_{PosTest} \in \mathcal{D}(2_T)$

# Bayesian Inference in Kleisli

**Inference question II** *What is the probability of a positive test **given*** that doctor I is **70% sure** the patient manifests the symptoms and doctor II is **50% sure** he doesn't?




$s \longmapsto 1$

$s^{\perp} \longmapsto 0$

1. Predicate *ObSy*: $2_S \to [0,1]$

2. Predicate transformation $S \ll ObSy : 2_D \to [0,1]$

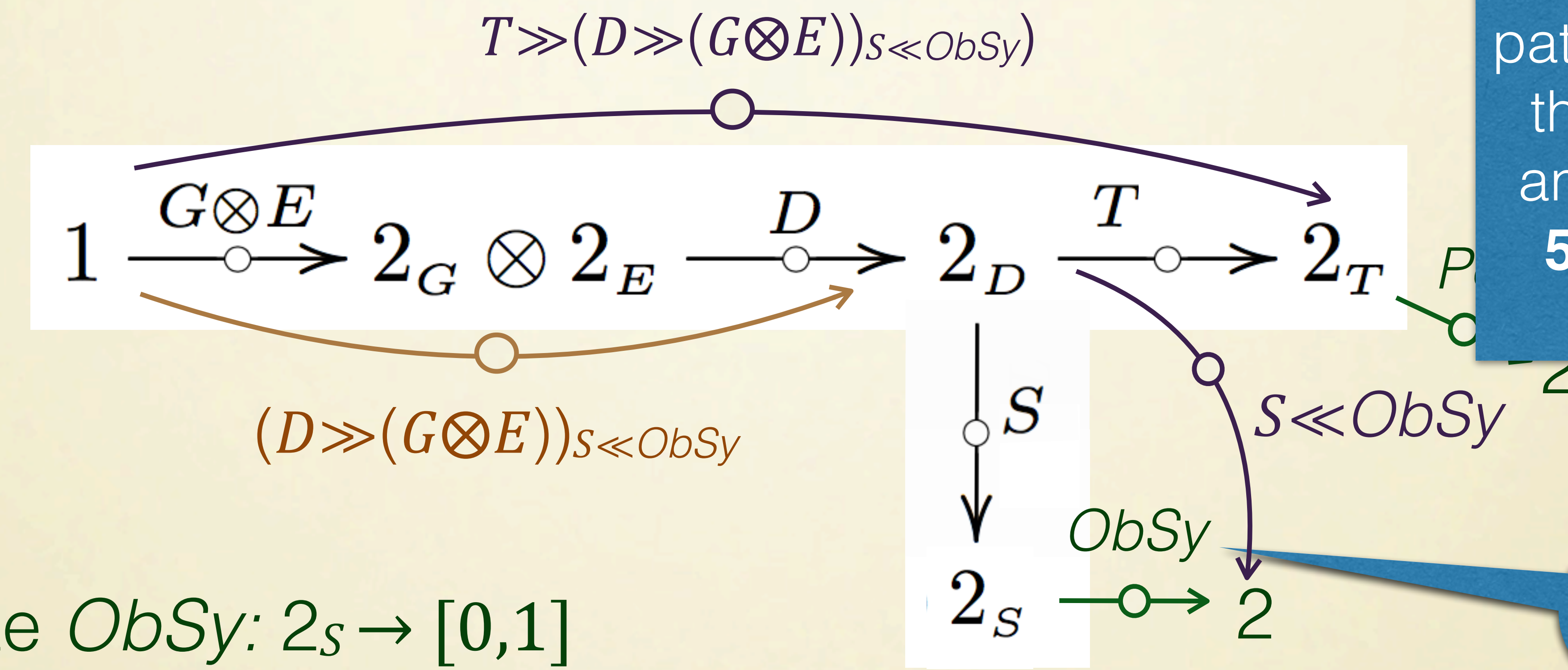
3. Conditioning $(D \gg (G \otimes E))_{S \ll ObSy} \in \mathcal{D}(2_D)$

4. State transformation $T \gg (D \gg (G \otimes E))_{S \ll ObSy}) \in \mathcal{D}(2_T)$

5. Answer is the conditioned state $(T \gg (D \gg (G \otimes E))_{S \ll ObSy}))_{PosTest} \in \mathcal{D}(2_T)$

# Bayesian Inference in Kleisli

**Inference question II** *What is the probability of a positive test* ***given*** that doctor I is **70% sure** the patient manifests the symptoms and doctor II is **50% sure** he doesn't?




$s \mapsto 0.7$
$s^{\perp} \mapsto 0.5$

1. Predicate *ObSy*: $2_S \rightarrow [0,1]$
2. Predicate transformation $S \ll ObSy : 2_D \rightarrow [0,1]$
3. Conditioning $(D \gg (G \otimes E))_{S \ll ObSy} \in \mathcal{D}(2_D)$
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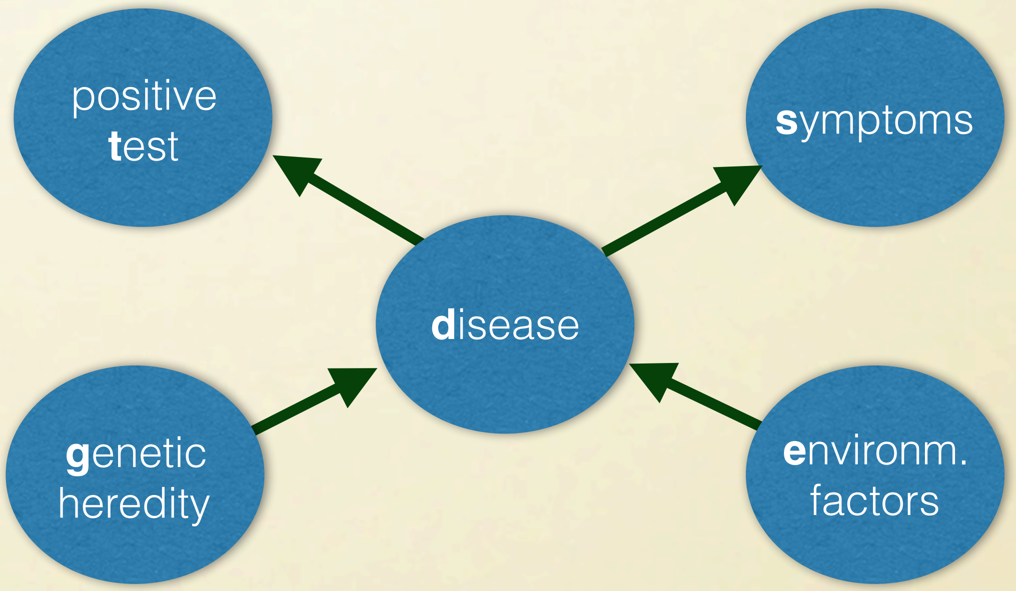
5. Answer is the conditioned state $(T \gg (D \gg (G \otimes E))_{S \ll ObSy}))_{PosTest} \in \mathcal{D}(2_T)$

# Influence

The two questions show that siblings can influence each others.



# Influence

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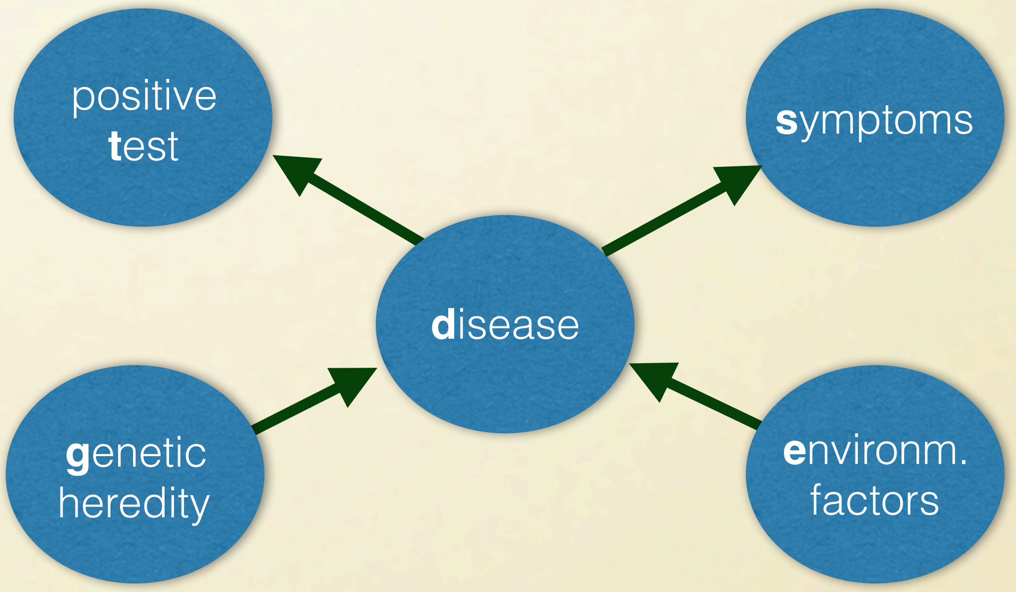
$T \gg D \gg (G \otimes E)$

*State*



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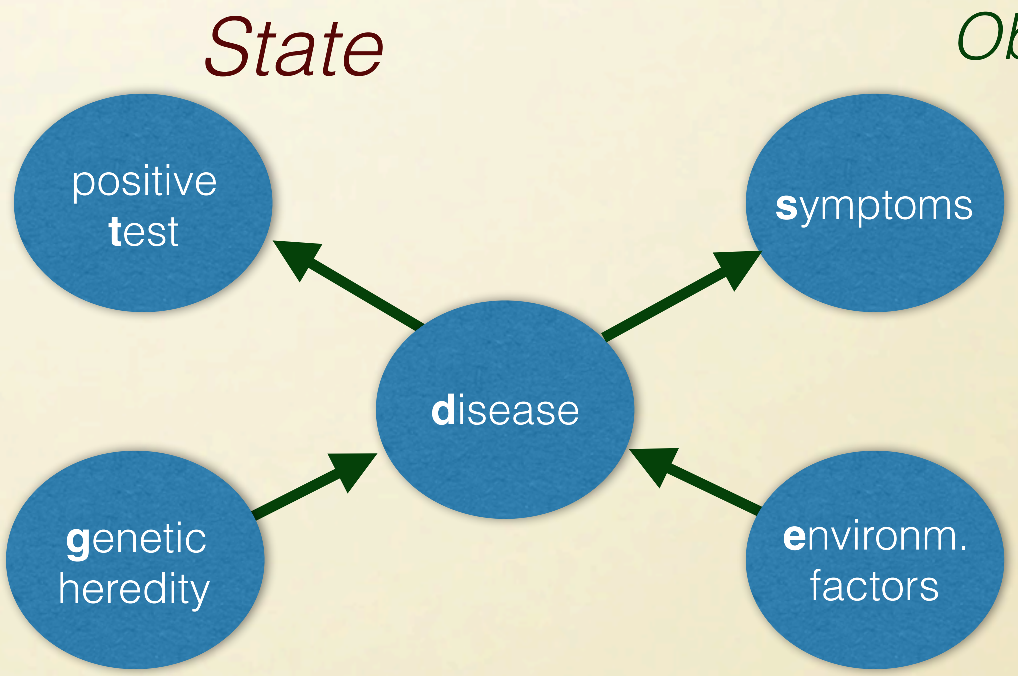
The two questions show that siblings can influence each others.

$$T \gg D \gg (G \otimes E)$$

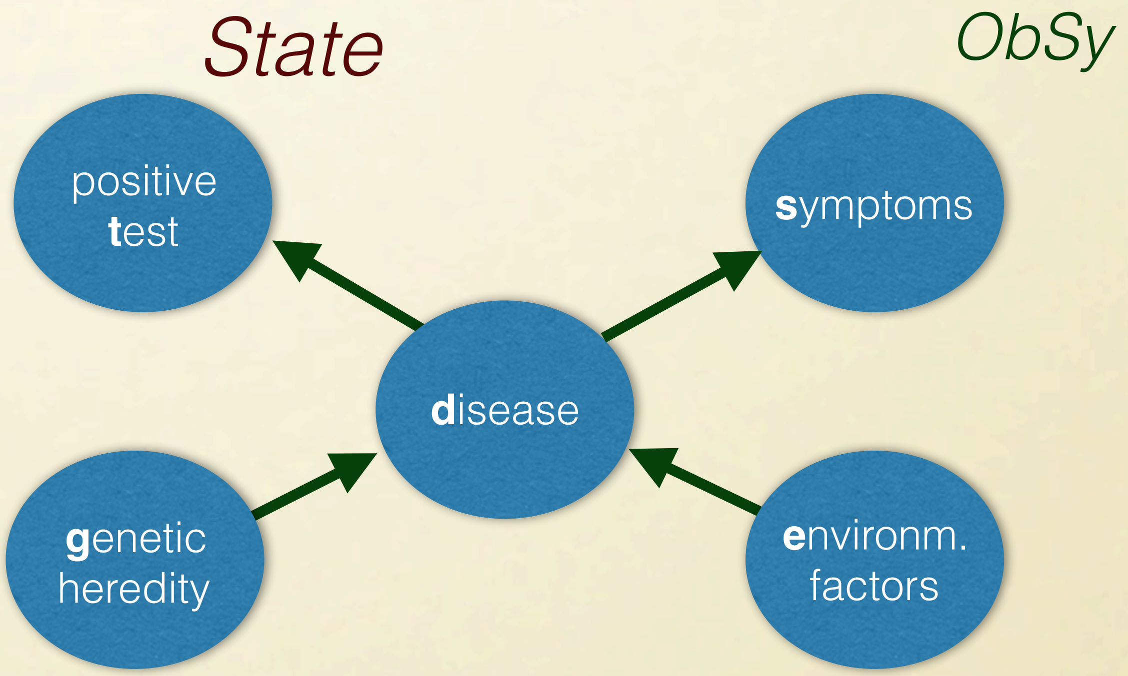
$$(T \gg (D \gg (G \otimes E))_{S \ll ObSy}))$$



# Influence

The two questions show that siblings can influence each others.

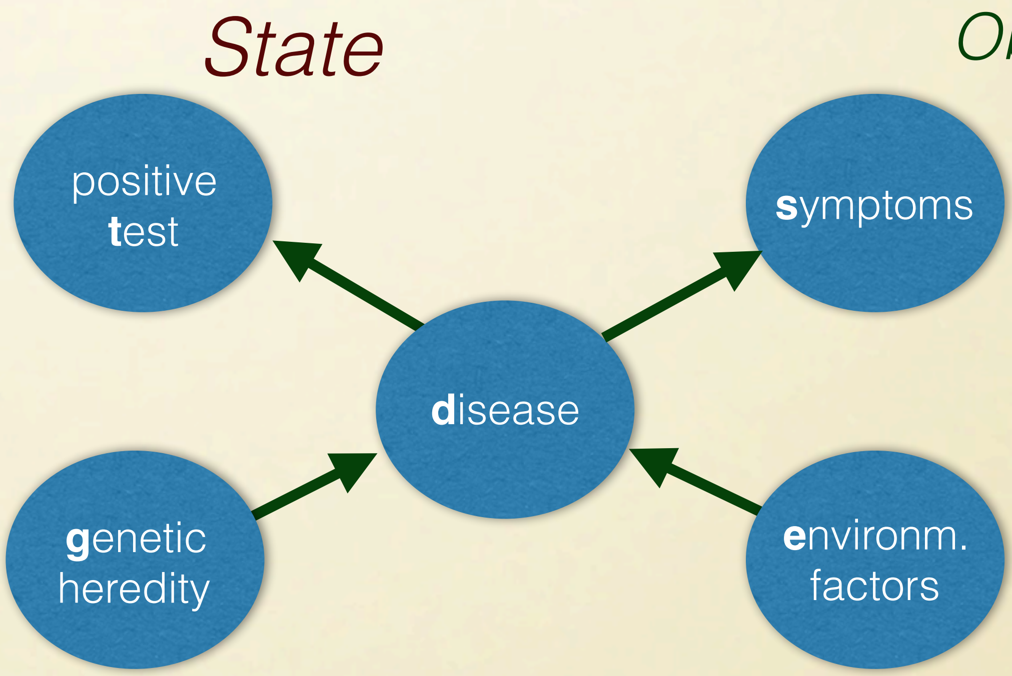
$$T \gg D \gg (G \otimes E)$$
$$\neq$$
$$(T \gg (D \gg (G \otimes E))_{S \ll ObSy}))$$



# Blocking Influence

But this influence can be **blocked.** The channel language is able to express and formally prove it.

$$T \gg D \gg (G \otimes E)$$
$$\neq$$
$$(T \gg (D \gg (G \otimes E))_{S \ll ObSy}))$$



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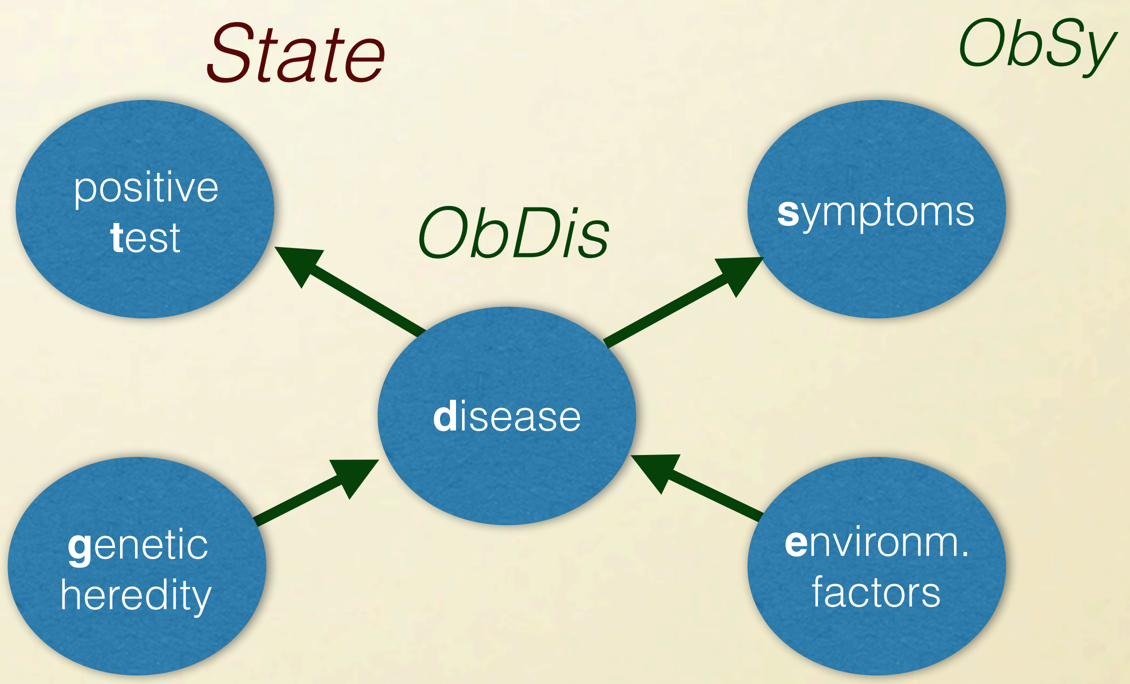
$$T \gg D \gg (G \otimes E) \neq (T \gg (D \gg (G \otimes E))_{S \ll ObSy}))$$

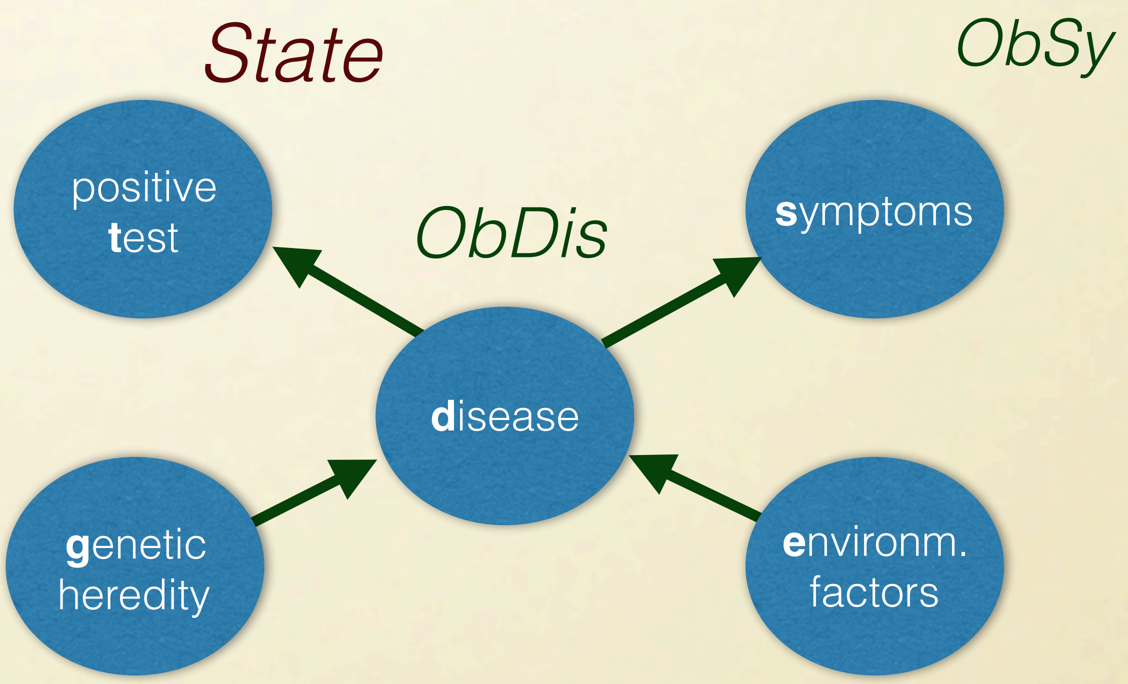


# Blocking Influence

But this influence can be **blocked.** The channel language is able to express and formally prove it.

$$T \gg D \gg (G \otimes E)$$
$$\neq$$
$$(T \gg (D \gg (G \otimes E))_{S \ll ObSy}))$$

$$T \gg (D \gg (G \otimes E))_{ObDis}$$
$$=$$
$$(T \gg ((D \gg (G \otimes E))_{ObDis})_{S \ll ObSy}))$$

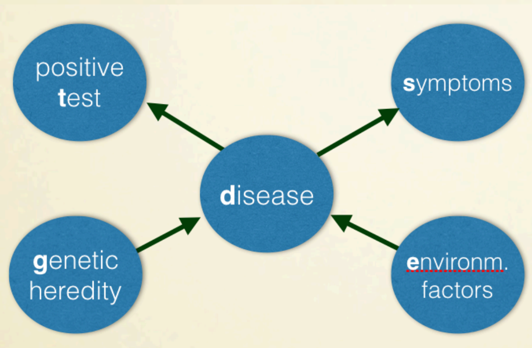


# Influence: overview

- More generally, the channel language allows to prove the three **d-separation** scenarios as formal statements.
- Influence can be formally quantified, via a (total variation) distance between states.

# Back to inference

Predicate/state transformation in **kl**($\boldsymbol{\mathcal{D}}$) offers a novel, dynamical style of performing Bayesian inference.




| g |
|---|
| 1/50 |

| e |
|---|
| 1/10 |

| | d |
|---|---|
| g e | 9/10 |
| g e$^\perp$ | 8/10 |
| g$^\perp$ e | 4/10 |
| g$^\perp$ e$^\perp$ | 0 |

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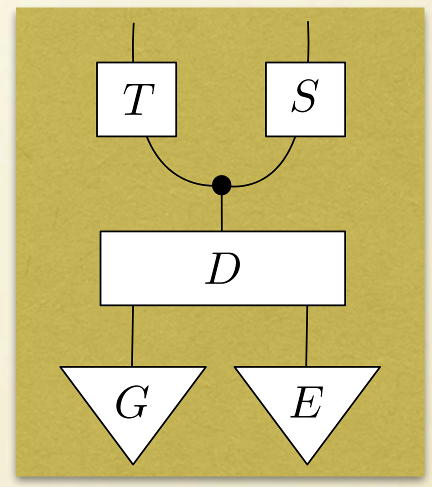
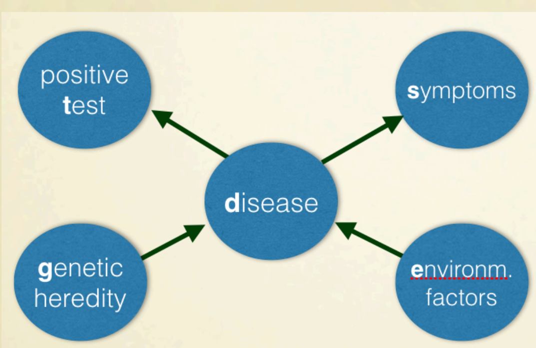
| d | 9/10 |
| d$^\perp$ | 1/20 |

| | s |
|---|---|
| d | 9/10 |
| d$^\perp$ | 1/15 |




In **kl**($\boldsymbol{\mathcal{D}}$) we can reproduce also a more traditional account of Bayesian inference, in which belief revision is performed on the whole joint distribution.





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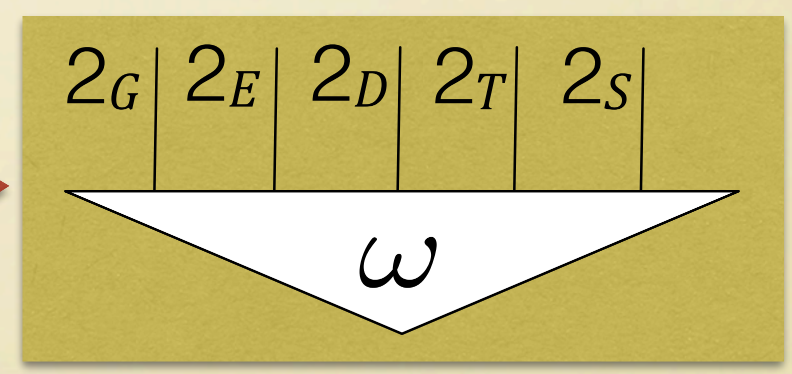
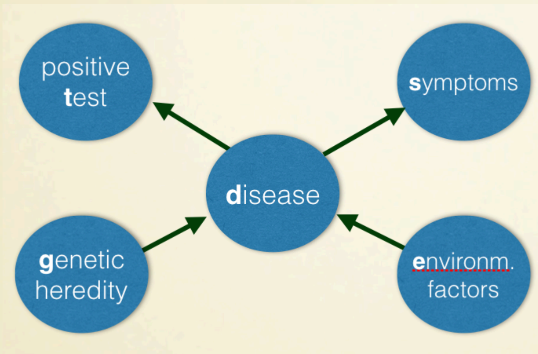


# Back to inference

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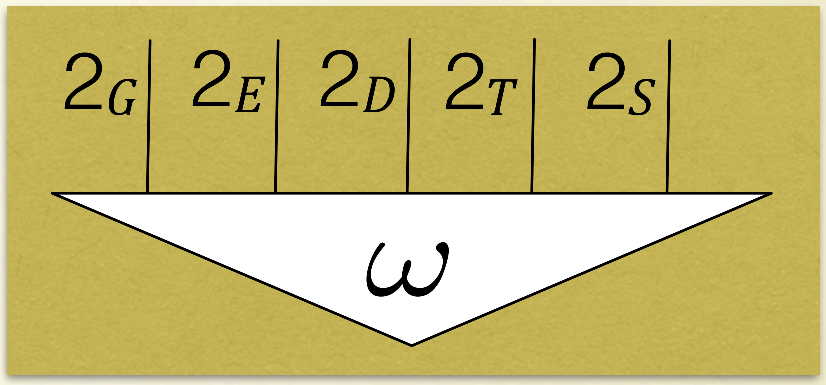



## Inference questions

*What is the a priori probability of a positive test?*

Answer:

$$M_4(\omega|_{(\mathrm{Id} \otimes \mathrm{Id} \otimes \mathrm{Id} \otimes \mathit{PosTest} \otimes \mathrm{Id})})$$

Fourth marginal (positive test node)

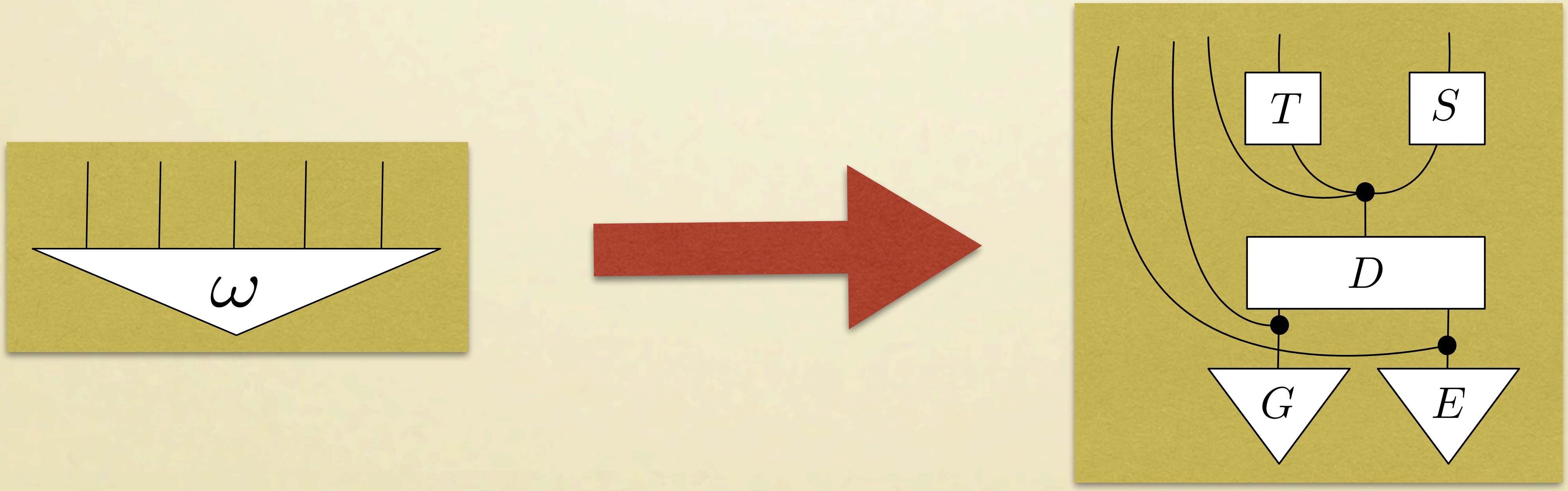
*What is the probability of a positive test **given** the symptoms?*

Answer:

$$M_4(\omega|_{(\mathrm{Id} \otimes \mathrm{Id} \otimes \mathrm{Id} \otimes \mathit{PosTest} \otimes \mathit{ObSy})})$$

# Back to inference

- It turns out the two styles of inference are provably **equivalent**.
- This can be made formally precise with **disintegration**: the process of factorising a given joint state into a Bayesian network.



# Bibliography

This talk:

- B. Jacobs and F. Zanasi - A Predicate/State Transformer Semantics for Bayesian Learning (proceedings of *MFPS* 2016)
- B. Jacobs and F. Zanasi - A Formal Semantics of Bayesian Influence (proceedings of *MFCS* 2017)
- B. Jacobs and F. Zanasi - The Logical Essentials of Bayesian Reasoning (Chapter in *Probabilistic Programming*, CUP, 2019)

Latest Developments:

- B. Jacobs, A. Kissinger, and F. Zanasi - Causal Inference via String Diagram Surgery (proceedings of *FOSSACS* 2019)
- B. Jacobs - Structured Probabilistic Reasoning (Book draft)

All of my papers are freely available at http://www.zanasi.com/fabio/#/publications.html