Contextuality: at the borders of paradox

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Contextuality

Renewed interest in contextuality in quantum information and foundations, as a **resource** in quantum information processing — perhaps the key one. (Howard, Wallman, Veith and Emerson, *Nature* 2014, Raussendorf PRA 2013).

Desiderata: analysis of **general concepts**, structural theory.
Disentangling contextuality from QM

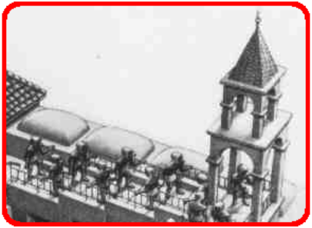
Our claims:

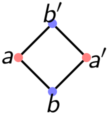
- Contextuality is a **general phenomenon**. It appears pervasively in many fields, e.g. logic and CS. Non-locality is a special case.
- There is a general structure and mathematical theory of contextuality, applicable across these fields.
- Contextual probability, Contextual semantics.

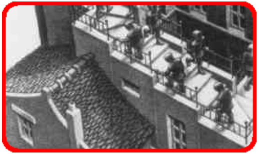
What **is** contextuality, as a problematic, non-classical phenomenon?

In a nutshell: where we have a family of data which is **locally consistent**, but **globally inconsistent**.

Contextuality Analogy: Local Consistency







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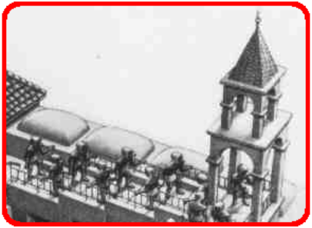
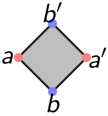
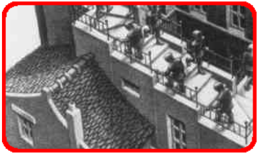
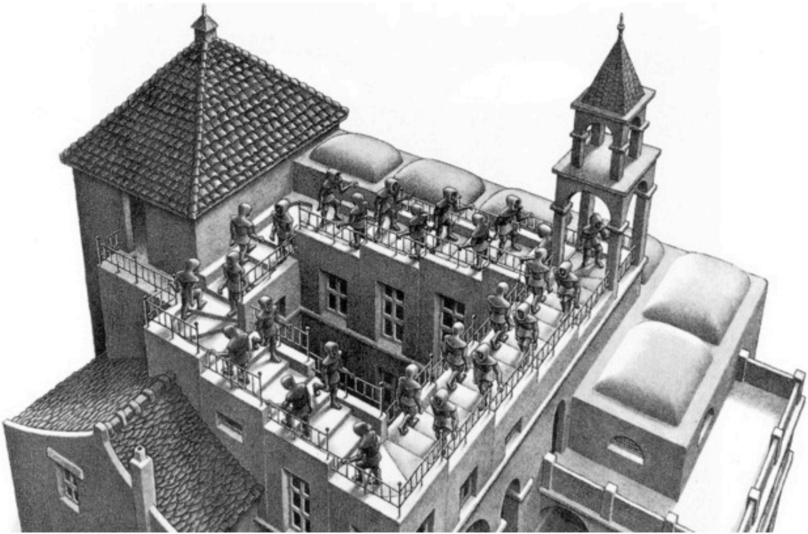






Contextuality Analogy: Global Inconsistency



The Borders of Paradox

If this phenomenon arises with **observable data**, reflecting physical reality, it takes us to the borders of paradox.

What saves us from a direct conflict between logic and experience is that the data **cannot** be directly observed globally.
We cannot observe all the variables **at the same time**.

A "transcendental deduction" of the **incompatibility** (in general) of observables.

The mathematics of contextuality

Thus contextuality is fundamentally about the passage from local to global, and **obstructions** to such a passage.

The natural mathematical language for talking about this is **sheaf theory**.

And we can use **sheaf cohomology** to witness contextuality.

Why is this important? Cohomology is one of the major tools of modern mathematics.

Has been conspicuously **absent** from this field (and indeed from Computer Science, logic, etc.)

Our results show that it does apply, in a very direct way, to the analysis of contextuality.

Quality as quantity

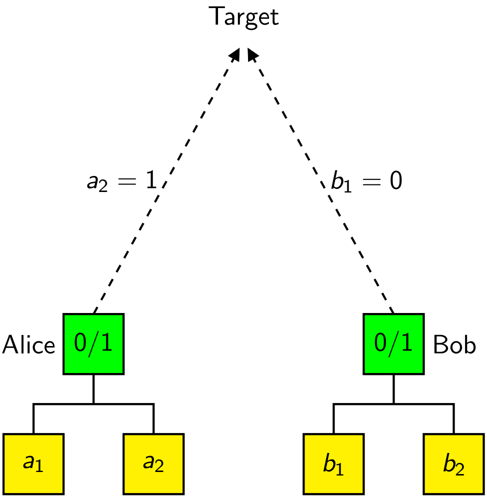
An objection: the real content of quantum mechanics involves probabilities, Bell inequalities, etc. . . .

But in fact, it turns out that there is a unifying principle for Bell inequalities based on logical consistency conditions.

In fact, **all Bell inequalities** arise from purely logical consistency conditions.

Logical and sheaf-theoretic structure also plays a key rôle in discerning a hierarchy of **degrees of contextuality**.

Alice and Bob look at bits



A Probabilistic Model Of An Experiment

Example: The Bell Model

A	B	$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$
a_1	b_1	$1/2$	0	0	$1/2$
a_1	b_2	$3/8$	$1/8$	$1/8$	$3/8$
a_2	b_1	$3/8$	$1/8$	$1/8$	$3/8$
a_2	b_2	$1/8$	$3/8$	$3/8$	$1/8$

The entry in row 2 column 3 says:

> *If Alice looks at* a_1 *and Bob looks at* b_2*, then* $1/8$*th of the time, Alice sees a* 0 *and Bob sees a* 1*.*

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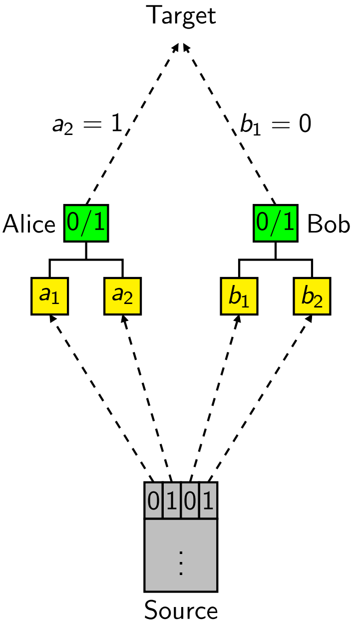
How can we explain this behaviour?

Classical Correlations: The Classical Source



A Simple Observation

Suppose we have propositional formulas $\phi_1, \ldots, \phi_N$

Suppose further we can assign a probability $p_i = \mathrm{Prob}(\phi_i)$ to each ϕ_i.

(Story: perform experiment to test the variables in ϕ_i; p_i is the relative frequency of the trials satisfying ϕ_i.)

Suppose that these formulas are **not simultaneously satisfiable**. Then (e.g.)

$$\bigwedge_{i=1}^{N-1} \phi_i \;\Rightarrow\; \neg\phi_N, \quad \text{or equivalently} \quad \phi_N \;\Rightarrow\; \bigvee_{i=1}^{N-1} \neg\phi_i.$$

Using elementary probability theory, we can calculate:

$$p_N \;\leq\; \mathrm{Prob}(\bigvee_{i=1}^{N-1} \neg\phi_i) \;\leq\; \sum_{i=1}^{N-1} \mathrm{Prob}(\neg\phi_i) \;=\; \sum_{i=1}^{N-1} (1 - p_i) \;=\; (N-1) - \sum_{i=1}^{N-1} p_i.$$

Hence we obtain the inequality

$$\sum_{i=1}^{N} p_i \;\leq\; N - 1.$$

Logical analysis of the Bell table

	$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$
(a_1, b_1)	**1/2**	0	0	**1/2**
(a_1, b_2)	**3/8**	1/8	1/8	**3/8**
(a_2, b_1)	**3/8**	1/8	1/8	**3/8**
(a_2, b_2)	1/8	**3/8**	**3/8**	1/8

If we read 0 as true and 1 as false, the highlighted entries in each row of the table are represented by the following propositions:

$$
\begin{array}{lclclclcl}
\varphi_1 & = & (a_1 \wedge b_1) & \vee & (\neg a_1 \wedge \neg b_1) & = & a_1 & \leftrightarrow & b_1 \\
\varphi_2 & = & (a_1 \wedge b_2) & \vee & (\neg a_1 \wedge \neg b_2) & = & a_1 & \leftrightarrow & b_2 \\
\varphi_3 & = & (a_2 \wedge b_1) & \vee & (\neg a_2 \wedge \neg b_1) & = & a_2 & \leftrightarrow & b_1 \\
\varphi_4 & = & (\neg a_2 \wedge b_2) & \vee & (a_2 \wedge \neg b_2) & = & a_2 & \oplus & b_2 .
\end{array}
$$

These propositions are easily seen to be contradictory.
The violation of the logical Bell inequality is $1/4$.

Example: the Hardy model

The support of the Hardy model:

	$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$
(a,b)	1	1	1	1
(a',b)	0	1	1	1
(a,b')	0	1	1	1
(a',b')	1	1	1	0

If we interpret outcome 0 as true and 1 as false, then the following formulas all have positive probability:

$$a \wedge b, \quad \neg(a \wedge b'), \quad \neg(a' \wedge b), \quad a' \vee b'.$$

However, these formulas are not simultaneously satisfiable.

In this model, $p_2 = p_3 = p_4 = 1$.

Hence the Hardy model achieves a violation of $p_1 = \text{Prob}(a \wedge b)$ for the logical Bell inequality.

A Possibilistic Model Of An Experiment

	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
(a_1, b_1)	1			
(a_1, b_2)	0			
(a_2, b_1)	0			
(a_2, b_2)				0

The entry in row 1 column 1 says:

> *If Alice looks at a_1 and Bob looks at b_1, then* **sometimes** *Alice sees a 0 and Bob sees a 0.*

The entry in row 2 column 1 says:

> *If Alice looks at a_1 and Bob looks at b_2, then* **it never happens** *that Alice sees a 0 and Bob sees a 0.*

Can we explain this behaviour using a classical source?

What Do 'Observables' Observe?

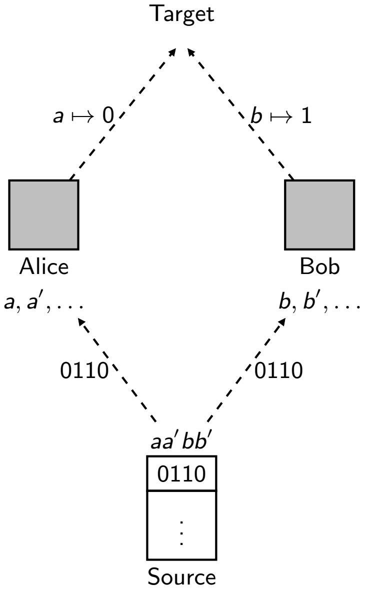
Surely **objective properties** of a physical system, which are independent of our choice of which measurements to perform — of our **measurement context**.

More precisely, this would say that for each possible state of the system, there is a function λ which for each measurement m specifies an outcome $\lambda(m)$, **independently of which other measurements may be performed**.

This point of view is called **non-contextuality**. It is equivalent to the assumption of a classical source.

However, this view is **impossible to sustain** in the light of our **actual observations of (micro)-physical reality**.

Hidden Variables: The Mermin instruction set picture



The 'Hardy Paradox'

Hardy models: those whose support satisfies

	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
(a_1, b_1)	1			
(a_1, b_2)	0			
(a_2, b_1)	0			
(a_2, b_2)				0

Which 'instruction set' λ could the outcomes $(0,0)$ for measurements (a_1, b_1) could come? Clearly, we must have

$$\lambda : a_1 \mapsto 0, \quad b_1 \mapsto 0.$$

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(a_1, b_2)	0	?		
(a_2, b_1)	0			
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So there is a unique 'instruction set' λ that outcomes $(0,0)$ for measurements (a_1, b_1) could come from:

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However, this would require the outcome $(0,0)$ for measurements (a_2, b_1) to be possible, and this is **precluded**.

Thus Hardy models are **contextual**. They cannot be explained by a classical source.

Quantum Mechanics changes the game

It seems then that the kind of behaviour exhibited in these tables is not realisable.

However, if we use **quantum** rather than classical resources, it **is** realisable!

More specifically, if we use an **entangled qubit** as a shared resource between Alice and Bob, who may be spacelike separated, then behaviour of exactly the kind we have considered **can** be achieved.

Alice and Bob's choices are now of **measurement setting** (e.g. which direction to measure spin) rather than "which register to load".

A Possibilistic Model Of An Experiment

	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
(a_1, b_1)	1			
(a_1, b_2)	0			
(a_2, b_1)	0			
(a_2, b_2)				0

This model can be **physically realised** in quantum mechanics.

There is an entangled state of two qubits, and directions for spin measurements a_1, a_2 for Alice and b_1, b_2 for Bob, which generate this table according to the predictions of quantum mechanics.

Moreover, behaviour of this kind has been extensively experimentally confirmed.

This is really how the world is!

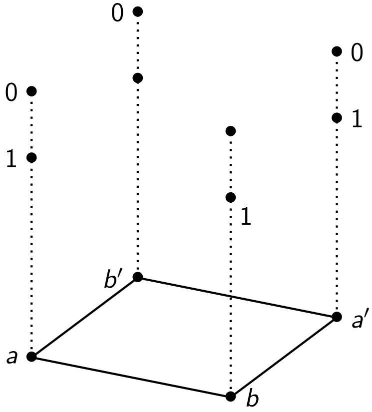
This proves a **strong version of Bell's theorem**.

Bundle Pictures

Logical Contextuality

- Ignore precise probabilities
- Events are possible or not
- E.g. the Hardy model:

	00	01	10	11
ab	✓	✓	✓	✓
ab'	×	✓	✓	✓
$a'b$	×	✓	✓	✓
$a'b'$	✓	✓	✓	×



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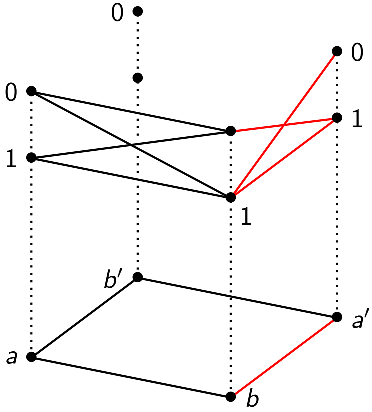

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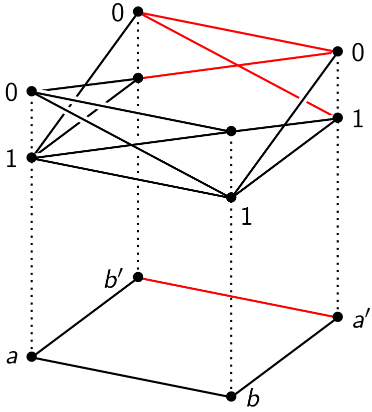

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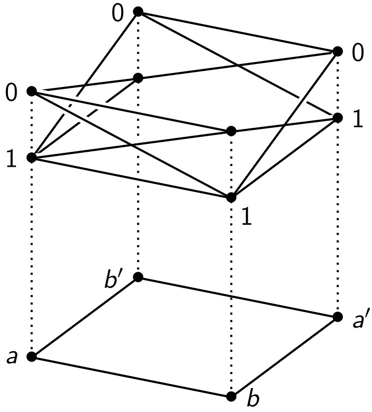

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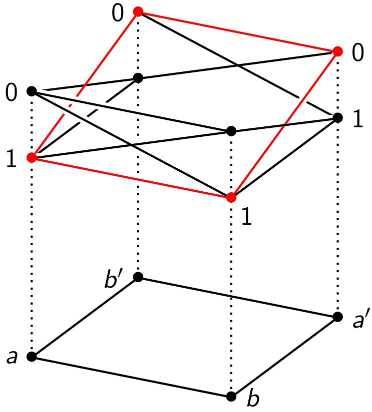

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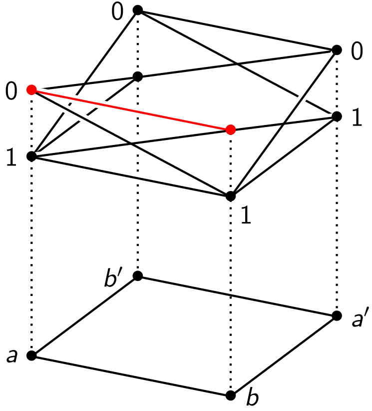

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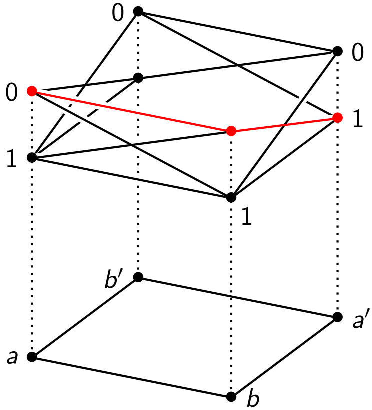

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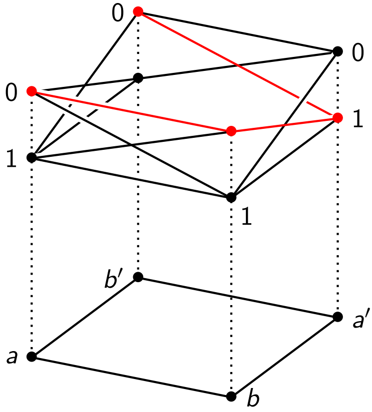

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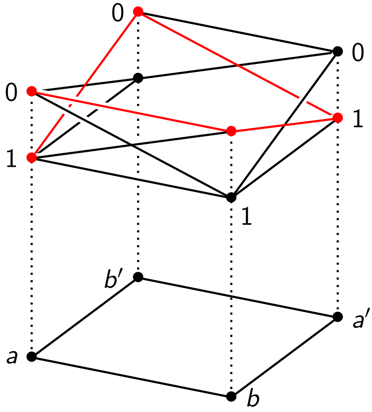

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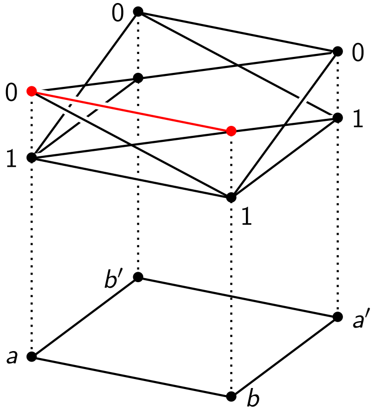

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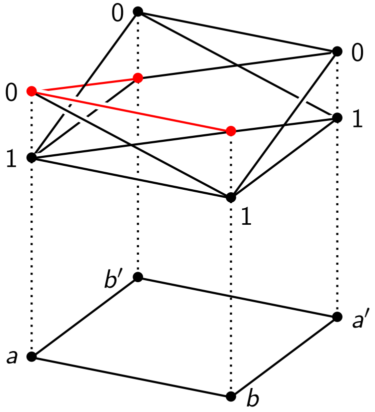

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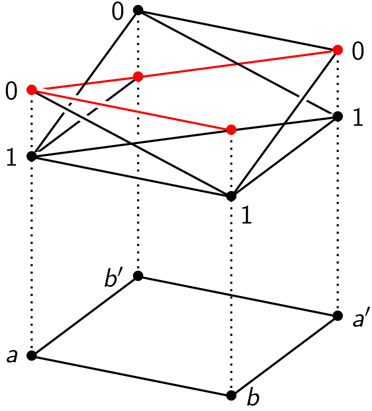

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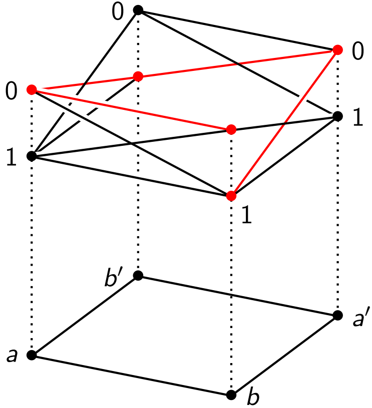
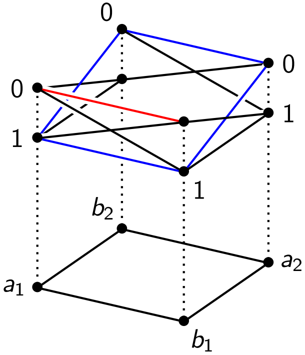


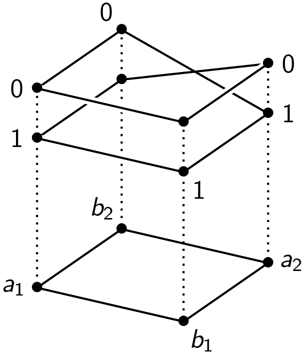
Strong Contextuality

A	B	$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$
a_1	b_1	1	0	0	1
a_1	b_2	1	0	0	1
a_2	b_1	1	0	0	1
a_2	b_2	0	1	1	0

The PR Box

Visualizing Contextuality





The Hardy table and the PR box as bundles

Contextuality, Logic and Paradoxes

Liar cycles. A Liar cycle of length N is a sequence of statements

$$\begin{aligned} S_1 &: \ S_2 \text{ is true,} \\ S_2 &: \ S_3 \text{ is true,} \\ &\vdots \\ S_{N-1} &: \ S_N \text{ is true,} \\ S_N &: \ S_1 \text{ is false.} \end{aligned}$$

For $N = 1$, this is the classic Liar sentence

$$S : \ S \text{ is false.}$$

Following Cook, Walicki et al. we can model the situation by boolean equations:

$$x_1 = x_2, \ \ \ldots, \ \ x_{n-1} = x_n, \ \ x_n = \neg x_1$$

The "paradoxical" nature of the original statements is now captured by the inconsistency of these equations.

Contextuality in the Liar; Liar cycles in the PR Box

We can regard each of these equations as fibered over the set of variables which occur in it:

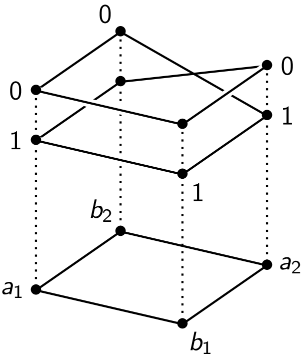
$$\begin{aligned}
\{x_1, x_2\} : \quad x_1 &= x_2 \\
\{x_2, x_3\} : \quad x_2 &= x_3 \\
&\vdots \\
\{x_{n-1}, x_n\} : \quad x_{n-1} &= x_n \\
\{x_n, x_1\} : \quad x_n &= \neg x_1
\end{aligned}$$

Any subset of up to $n-1$ of these equations is consistent; while the whole set is inconsistent.

Up to rearrangement, **the Liar cycle of length 4 corresponds exactly to the PR box**.

The usual reasoning to derive a contradiction from the Liar cycle corresponds precisely to the attempt to find a univocal path in the bundle diagram.

Paths to contradiction



Suppose that we try to set a_2 to 1. Following the path on the right leads to the following local propagation of values:

$$a_2 = 1 \rightsquigarrow b_1 = 1 \rightsquigarrow a_1 = 1 \rightsquigarrow b_2 = 1 \rightsquigarrow a_2 = 0$$
$$a_2 = 0 \rightsquigarrow b_1 = 0 \rightsquigarrow a_1 = 0 \rightsquigarrow b_2 = 0 \rightsquigarrow a_2 = 1$$

We have discussed a specific case here, but the analysis can be generalised to a large class of examples.

The Robinson Consistency Theorem

A classic result:

Theorem (Robinson Joint Consistency Theorem)

Let T_i be a theory over the language L_i, $i = 1, 2$. If there is no sentence ϕ in $L_1 \cap L_2$ with $T_1 \vdash \phi$ and $T_2 \vdash \neg\phi$, then $T_1 \cup T_2$ is consistent.

Thus this theorem says that two compatible theories can be glued together. In this binary case, local consistency implies global consistency.

Note, however, that an extension of the theorem beyond the binary case **fails**. That is, if we have three theories which are pairwise compatible, it need not be the case that they can be glued together consistently.

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A minimal counter-example is provided at the propositional level by the following "triangle":

$$T_1 = \{x_1 \longrightarrow \neg x_2\}, \; T_2 = \{x_2 \longrightarrow \neg x_3\}, \; T_3 = \{x_3 \longrightarrow \neg x_1\}.$$

This example is well-known in the quantum contextuality literature as the **Specker triangle**.

Sheaf formulation of contextuality

Measurement scenarios $\langle X, \mathcal{M}, O \rangle$:

- X is a set of variables or measurement labels. Sufficient to consider finite discrete space — the base space of the bundle.
- $\mathcal{M} = \{C_i\}_{i \in I}$ set of **contexts** *i.e.* co-measurable variables. In quantum terms, compatible observables.
- O is set of outcomes or values for the variables, which we take to be the same in each fibre.

We have a sheaf of sets over $\mathcal{P}(X)$, namely $\mathcal{E} :: U \longmapsto O^U$ with restriction

$$\mathcal{E}(U \subseteq U') \colon \mathcal{E}(U') \longrightarrow \mathcal{E}(U) :: s \longmapsto s|U \ .$$

Each $s \in \mathcal{E}(U)$ is a **section**, and, in particular, $g \in \mathcal{E}(X)$ is a **global section**.

A probability table can be represented by a family $\{p_C\}_{C \in \mathcal{M}}$ with p_C a probability distribution on $\mathcal{E}(C) = O^C$, where contexts C corresponds to the rows of the table.

Empirical Models

The logical and strong forms of contextuality are concerned with **possibilities**, which can be represented by a subpresheaf $\mathcal{S}$ of $\mathcal{E}$, where for each context $U \subseteq X$, $\mathcal{S}(U) \subseteq O^U$ is the set of all possible outcomes.

Explicitly, $\mathcal{S}$ is defined as follows, where $\mathsf{supp}\,(p_C|U \cap C)$ is the support of the marginal of p_C at $U \cap C$.

$$\mathcal{S}(U) := \left\{ s \in O^U \;\middle|\; \forall C \in \mathcal{M}.\ s|U \cap C \in \mathsf{supp}\,(p_C|U \cap C) \right\}$$

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We can use this formalisation to characterize contextuality as follows.

Definition

For any empirical model $\mathcal{S}$:

- For all $C \in \mathcal{M}$ and $s \in \mathcal{S}(C)$, $\mathcal{S}$ is logically contextual at s, written $\mathsf{LC}(\mathcal{S}, s)$, if s is not a member of any compatible family.
- $\mathcal{S}$ is **strongly contextual**, written $\mathsf{SC}(\mathcal{S})$, if $\mathsf{LC}(\mathcal{S}, s)$ for all s. Equivalently, if it has no global section, *i.e.* if $\mathcal{S}(X) = \varnothing$.

Summary of Cohomological Characterization

We have a cover

$$\mathcal{U} = \{C_1, \ldots, C_n\}$$

of measurement contexts.

Given $s = s_1 \in S_e(C_1)$, we define

$$z = \delta^0(s_1, \ldots, s_n),$$

where $s_1|_{C_1 \cap C_i} = s_i|_{C_1 \cap C_i}$, $i = 1, \ldots, n$.

This is a cocycle in the relative Čech cohomology with respect to C_1.

We define

$$\gamma(s) = [z] \in \check{H}^1(\mathcal{U}, \mathcal{F}_{\bar{C}_1})$$

where $\mathcal{F}$ is the **AbGrp**-valued presheaf $\mathbb{Z}[S_e]$.

Here γ is in fact the **connecting homomorphism** of the long exact sequence.

Basic Results

Proposition

The following are equivalent:

1. *The cohomology obstruction vanishes:* $\gamma(s_1) = 0$.
2. *There is a family* $\{r_i \in \mathcal{F}(C_i)\}$ *with* $s_1 = r_1$, *and for all* i, j*:*

$$r_i | C_i \cap C_j = r_j | C_i \cap C_j.$$

Proposition

If the model e is possibilistically extendable, then the obstruction vanishes for every section in the support of the model. If e is not strongly contextual, then the obstruction vanishes for some section in the support.

Thus non-vanishing of the obstruction provides a cohomological witness for contextuality.

Notes on Cohomology

- There are false positives because of negative coefficients in cochains.
- We can effectively compute (mod 2) witnesses in many cases of interest: GHZ, Kylachko, Peres-Mermin, large class of Kochen-Specker models, . . .
- In recent work, we obtain very general results in cases where the outcomes themselves have a module structure (over the same ring as the cohomology coefficients).

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- In recent work, we obtain very general results in cases where the outcomes themselves have a module structure (over the same ring as the cohomology coefficients).
- This yields cohomological characterisations of **All-vs.-Nothing** proofs (Mermin). These account for most of the contextuality arguments in the quantum literature. In particular, we can find large classes of concrete examples in **stabiliser QM**.

Theorem

Let $\mathcal{S}$ be an empirical model on $\langle X, \mathcal{M}, R\rangle$. Then:

$$\mathsf{AvN}_R(\mathcal{S}) \;\Rightarrow\; \mathsf{SC}(\mathsf{Aff}\,\mathcal{S}) \;\Rightarrow\; \mathsf{CSC}_R(\mathcal{S}) \;\Rightarrow\; \mathsf{CSC}_{\mathbb{Z}}(\mathcal{S}) \;\Rightarrow\; \mathsf{SC}(\mathcal{S}) \;.$$

Relational databases

This geometric picture and the associated methods can be applied to a wide range of situations in classical computer science.

In particular, as we shall now see, there is an isomorphism between the formal description we have given for the quantum notions of non-locality and contextuality, and basic definitions and concepts in relational database theory.

Samson Abramsky, 'Relational databases and Bell's theorem', In *In Search of Elegance in the Theory and Practice of Computation: Essays Dedicated to Peter Buneman*, Springer 2013.

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Cambridge	10991-06284	Newton	£2,567.53
Hanover	10992-35671	Leibniz	€11,245.75
...	...	...	...

From possibility models to databases

Consider again the Hardy model:

	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
(a_1, b_1)	1	1	1	1
(a_1, b_2)	0	1	1	1
(a_2, b_1)	0	1	1	1
(a_2, b_2)	1	1	1	0

Change of perspective:

$a_1,\ a_2,\ b_1,\ b_2$	attributes
0, 1	data values
joint outcomes of measurements	tuples

The Hardy model as a relational database

The four rows of the model turn into four **relation tables**:

a_1	b_1
0	0
0	1
1	0
1	1

a_1	b_2
0	1
1	0
1	1

a_2	b_1
0	1
1	0
1	1

a_2	b_2
0	0
1	0
0	1

What is the DB property corresponding to the presence of non-locality/contextuality in the Hardy table?

There is no **universal relation**: no table

a_1	a_2	b_1	b_2
$\vdots$	$\vdots$	$\vdots$	$\vdots$

whose projections onto $\{a_i, b_i\}$, $i = 1, 2$, yield the above four tables.

A dictionary

Relational databases	measurement scenarios
attribute	measurement
set of attributes defining a relation table	compatible set of measurements
database schema	measurement cover
tuple	local section (joint outcome)
relation/set of tuples	boolean distribution on joint outcomes
universal relation instance	global section/hidden variable model
acyclicity	Vorob'ev condition

We can also consider probabilistic databases and other generalisations; cf. provenance semirings.

Contextual Semantics

Why do such similar structures arise in such apparently different settings?

The phenomenon of contextuality is pervasive. Once we start looking for it, we can find it everywhere!
Physics, computation, logic, natural language, . . . biology, economics, . . .

The **Contextual semantics hypothesis**: we can find common mathematical structure in all these diverse manifestations, and develop a widely applicable theory.

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For an accessible overview of Contextual Semantics, see the article in the *Logic in Computer Science* Column, Bulletin of EATCS No. 113, June 2014 (and `arXiv`).

People

Comrades in Arms in Contextual Semantics:










Adam Brandenburger, Lucien Hardy, Shane Mansfield, Rui Soares Barbosa, Ray Lal, Mehrnoosh Sadrzadeh, Phokion Kolaitis, Georg Gottlob, Carmen Constantin, Kohei Kishida

Some Recent Developments

- **Hardy is almost everywhere**: with bipartite exceptions, an algorithm which given an n-qubit entangled state, constructs $n + 2$ local observables leading to a logically contextual model.
- Characterization of the **face lattice** of the No-Signalling polytope as isomorphic to the support lattice.
- General characterisation of **All-versus-Nothing** arguments. Use of **sheaf cohomology** to capture contextuality for all such models. Large classes of quantum examples using stabiliser groups.

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